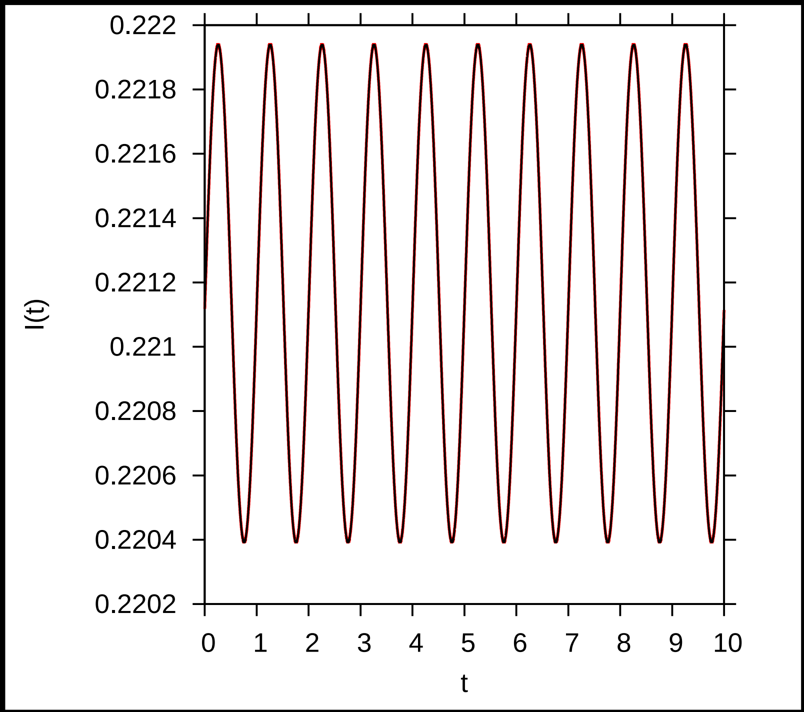
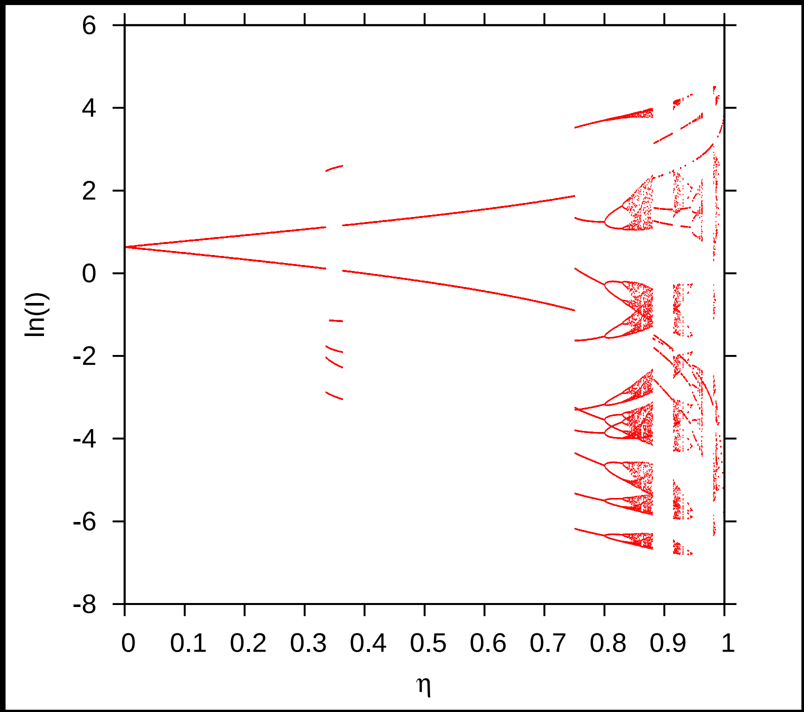
The role of seasonality in vector-borne disease dynamics





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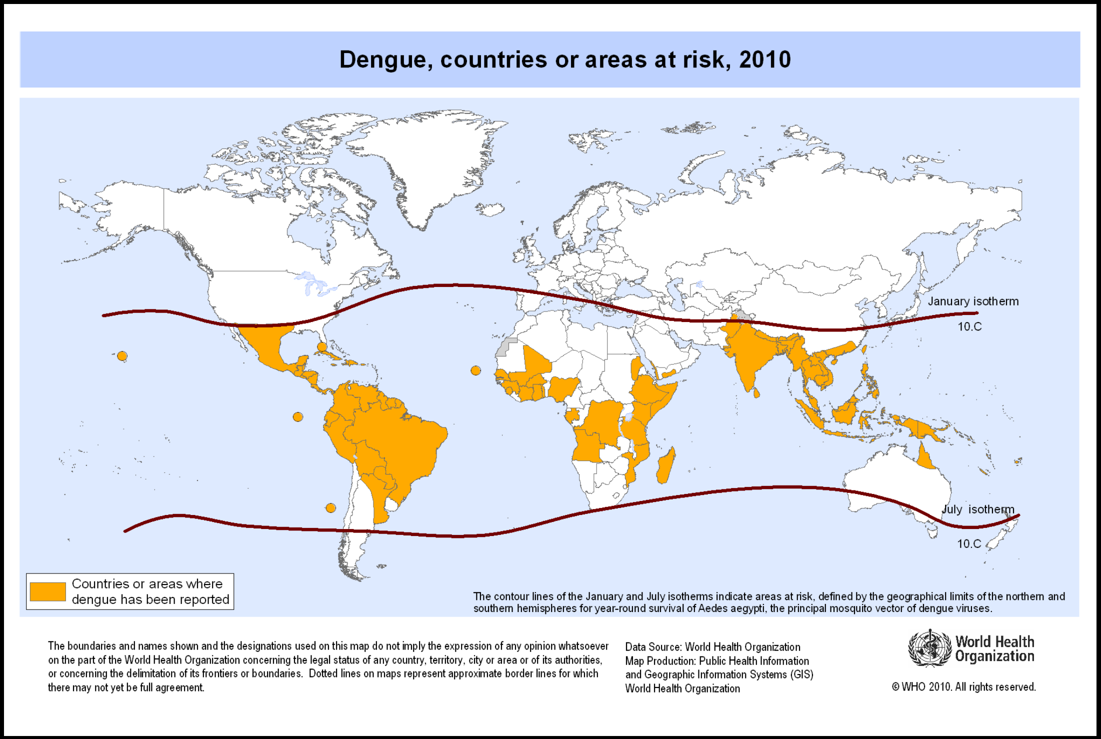
Dengue Fever Epidemiology

Dengue is a mosquito-borne infection caused by an arbivirus with 4 serotypes DENV1-4.

The distribution is in tropical and subtropical areas. However, the disease is spreanding to northern sites, being in the ”gates of Europe”.

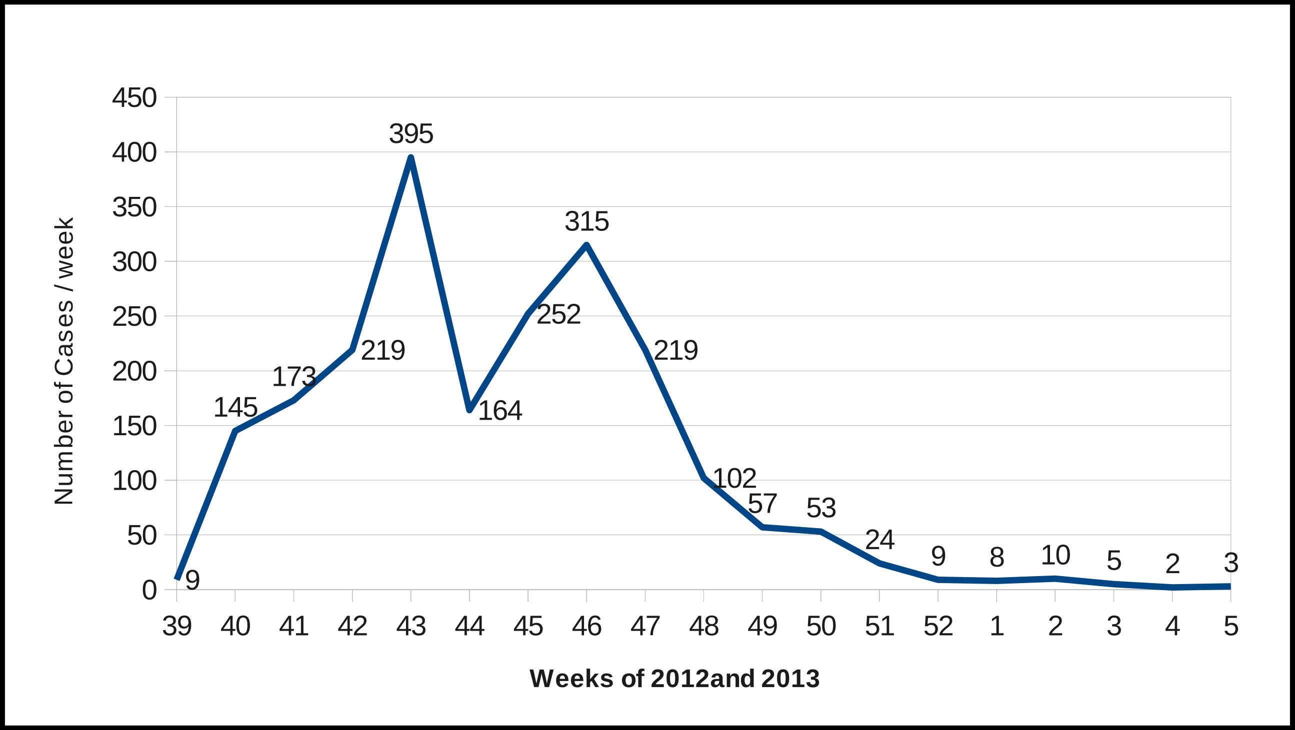
Recently the disease arrived to Madeira island and there is an outbreak with more than 2000 cases.



Worldwide dengue distribution in 2010 and areas at risk - Source: WHO (2012)

Dengue in Madeira

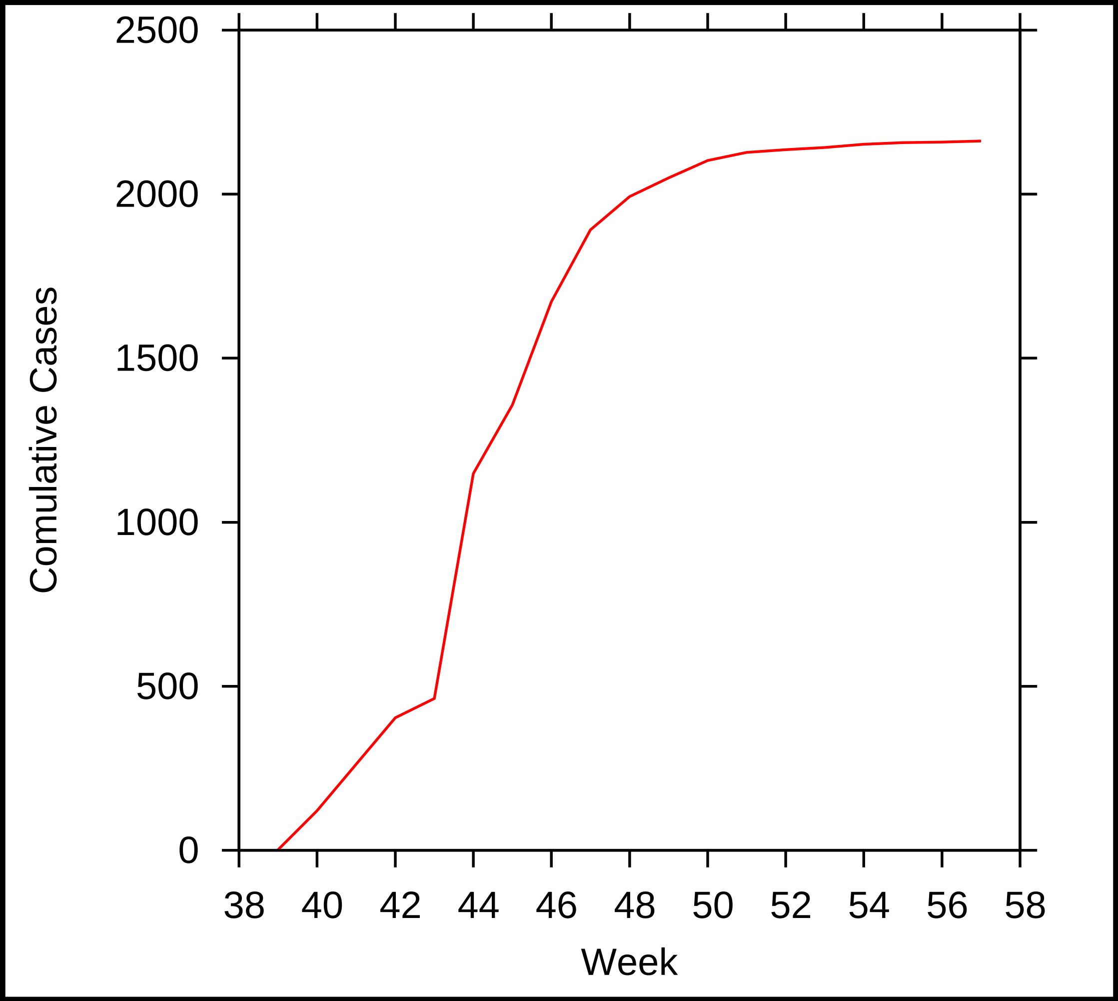
The outbreak started in early Automn season and has been developing.



The virus seem to be the DENV-1, and people think that the disease was imported from the Americas (Brazil or Venezuela).

Dengue in Madeira

Comulatively it has been reported 2164 cases.



Dengue in Madeira

The outbreak is mainly in Funchal, but the disease is spreading through the island and surounding islands.

Moreover, 78 cases of infected people were exported from the archipelago. Mainly in people from Portugal, but also from other countries such as UK, Germany, Sweden, France and Finland.

The vector

The main vector is the *Aedes aegypti*, original from Africa, is now more distributed in Americas.

This species has been identified in Madeira island since 2005.

Other vector species is the *Aedes albopictus*, which is more distributed in Asia, Northern Africa and Europe.

This species has been identified to Spain, France, Italy, Croatia, Greece, between others.

Usually is verified an increase in number of mosquitos during the warmer seasons, spetially in temperate regions.

Time-scale separation in SISUV

The simplified version of the SISUV model, considering constant population size for human and mosquitos, is

$$\frac{d}{dt}I = \frac{\beta}{M}(N - I)V - \alpha I$$

$$\frac{d}{dt}V = \frac{\vartheta}{N}(M - V)I - \nu V$$

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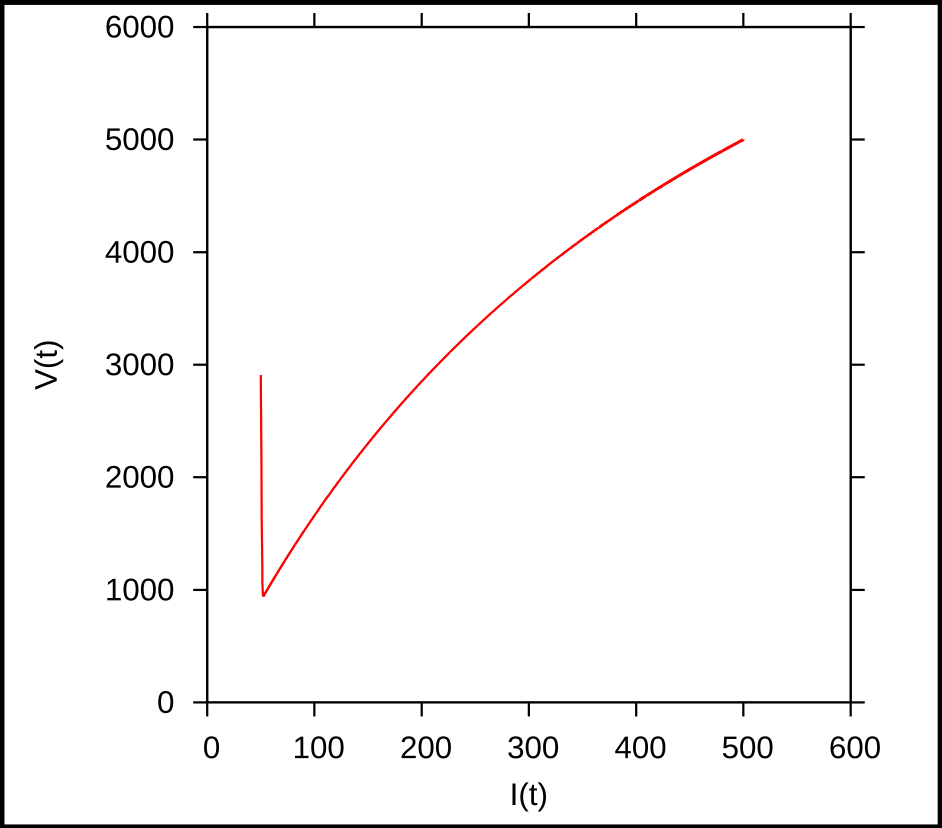
However it is noticed that the mosquitos' dynamics is faster than the humans', so we modified the variables os mosquitos dynamics $\left(\vartheta =: \frac{\bar{\vartheta}}{\varepsilon} \quad \text{and} \quad \nu =: \frac{\bar{\nu}}{\varepsilon}\right)$ **in order to put them in the same range of the human's.**

$$\frac{d}{dt}I = \frac{\beta}{M}(N-I)V - \alpha I$$

$$\frac{d}{dt}V = \frac{1}{\varepsilon}\left(\frac{\bar{\vartheta}}{N}(M-V)I - \bar{\nu}V\right)$$

Time-scale separation in SISUV



Using the following parameters set:

$$\alpha = \frac{1}{10\ \mathrm{y}},\ \beta = 2 \cdot \alpha,\ \nu = \frac{1}{10\ \mathrm{d}} = \frac{365}{10}\mathrm{y}^{-1} \text{ and } \vartheta = 2 \cdot \nu\ .$$

Time-scale separation in SISUV

Considering the normal time scale given by t and the fast time scale given by $\tau := \frac{t}{\varepsilon}$, the general solution for the ODE system is:

$$
\begin{aligned}
I &= I_0 + \varepsilon I_1 + \varepsilon^2 I_2 + \mathcal{O}(\varepsilon^3) \\
V &= V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \mathcal{O}(\varepsilon^3)
\end{aligned}
$$

And for the slow time scale we obtain from the right hand side of the ODE system

$$
\frac{dI}{dt} = \varepsilon^0 \left(\frac{\beta}{M} (N V_0 - I_0 V_0) - \alpha I_0 \right) + \varepsilon^1 \left(\frac{\beta}{M} (N V_1 - I_1 V_0 - I_0 V_1) - \alpha I_1 \right) + \mathcal{O}(\varepsilon^2)
$$

$$
\frac{dV}{dt} = \frac{1}{\varepsilon} \left(\frac{\bar{\vartheta}}{N} (M I_0 - V_0 I_0) - \bar{\nu} V_0 \right) + \varepsilon^0 \left(\frac{\bar{\vartheta}}{N} (M I_1 + V_1 I_0 + V_0 I_1) - \bar{\nu} V_1 \right) + \mathcal{O}(\varepsilon^1)
$$

Time-scale separation in SISUV

Being $\frac{dI}{d\tau} = \varepsilon\frac{dI}{dt}$ and $\frac{dV}{d\tau} = \varepsilon\frac{dV}{dt}$, if we substitute on the right hand of the ODEs we obtain

$$\frac{dI}{d\tau} = \underbrace{\varepsilon\left(\frac{\beta}{M}\left(N - I_0\right)V_0 - \alpha I_0\right)}_{=\frac{dI_0}{d\tau}} + \mathcal{O}(\varepsilon^2)$$

$$\frac{dV}{d\tau} = \underbrace{\varepsilon^0\left(\frac{\bar{\vartheta}}{N}\left(M - V_0\right)I_0 - \bar{\nu}V_0\right)}_{=\frac{dV_0}{d\tau}} + \mathcal{O}(\varepsilon^1)$$

Or, for exactly $\varepsilon = 0$, the derivatives are:

$$\frac{dI_0}{d\tau} = 0$$

$$\frac{dV_0}{d\tau} = \left(\frac{\bar{\vartheta}}{N}\left(M - V_0\right)I_0 - \bar{\nu}V_0\right)$$

Time-scale separation in SISUV

As the infected has not fast time-scale, so $\frac{dI_0}{d\tau} = 0$ and all values of $I_0(\tau) = I_0(\tau_0)$. So, substituting $I_0(\tau)$ in $\frac{dV_0}{d\tau}$ it is obtained:

$$\frac{dV_0}{d\tau} = -\left(\frac{\bar{\vartheta}}{N} I_0(\tau_0) + \bar{\nu}\right) V_0 + \frac{\bar{\vartheta}}{N} M I_0(\tau_0)$$

Which approaches very rapidly in an exponential way to its local stationary state:

$$V_0^* = \frac{\frac{\bar{\vartheta}}{N} I_0(\tau_0)}{\frac{\bar{\vartheta}}{N} I_0(\tau_0) + \bar{\nu}} \cdot M$$

Time-scale separation in SISUV

Now to the slow dynamics:

$$\frac{dI_0}{dt} = \left(\frac{\beta}{M}(NV_0 - I_0V_0) - \alpha I_0\right)$$

$$\varepsilon\,\frac{dV_0}{dt} = \left(\frac{\bar{\vartheta}}{N}(MI_0 - V_0I_0) - \nu V_0\right)$$

If we set $\varepsilon = 0$**, we can obtain the equation of** $V_0(t)$**, for any time** t**:**

$$V_0(t) = \frac{\frac{\bar{\vartheta}}{N}I_0(t)}{\frac{\bar{\vartheta}}{N}I_0(t) + \bar{\nu}} \cdot M$$

And now, finally, we can find the global stationary state:

$$I^* = \frac{\beta - \alpha \cdot \frac{\nu}{\vartheta}}{\beta + \alpha}N \quad \text{and} \quad V^* = \frac{\beta - \alpha \cdot \frac{\nu}{\vartheta}}{\beta\left(1 + \frac{\nu}{\vartheta}\right)}\,M$$

Time-scale separation in SISUV

The Jacobian matrix of the model is given by:

$$A = \begin{pmatrix} -\frac{\beta}{M} \cdot V^* - \alpha & \frac{\beta}{M} \cdot (N - I^*) \\ \frac{\bar{\vartheta}}{\varepsilon N} \cdot (M - V^*) & \frac{1}{\varepsilon}\left(-\frac{\bar{\vartheta}}{N} \cdot I^* - \bar{\nu}\right) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The eigenvalues of are given by:

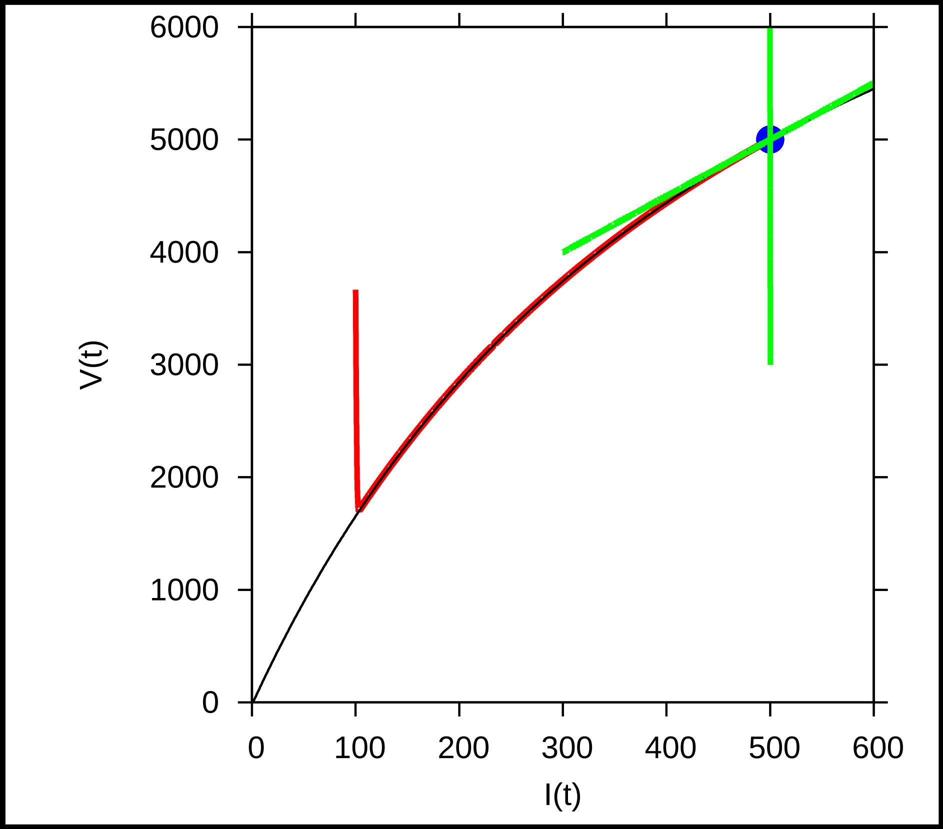
$$\lambda_{1/2} = \frac{(a+d)}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - (ad - bc)}$$

And the numerical simualtions shows that one is close to 0 and the other is large negative ($\lambda_1 = 0$ and $\lambda_2 = -73$).

And the general formula of eigenvectors is:

$$\underline{u}_i = \frac{1}{\sqrt{1 + \left(\frac{c}{d-\lambda_i}\right)^2}} \begin{pmatrix} 1 \\ -\frac{c}{d-\lambda_i} \end{pmatrix}$$

Time-scale separation in SISUV



Center manifold analysis in SISUV

Start by shifting the system (I, V) into a (z, w) system with the endemic fixed point at the origin:

$$\begin{aligned} z &:= I - I^* \\ w &:= V - V^* \end{aligned}$$

Rearranging the system and considering the non-trivial stationary state as the origin of a (x, y) system and the eigendirections as coordinate axis. This transformation is done considering:

$$\underline{x} := T^{-1}\underline{z}$$

Substituting:

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k & 0 \\ \frac{c}{d} & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} kz \\ \frac{c}{d}z + w \end{pmatrix}$$

Similarly, it is possible to calculate $\underline{z}$:

$$\underline{z} = \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & 0 \\ -\frac{c}{d}\frac{1}{k} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{k}x \\ -\frac{c}{d}\frac{1}{k}x + y \end{pmatrix}$$

Center manifold analysis in SISUV

The ODE system from the original (I, V) to the $\underline{z}$ system is given by $\frac{d}{dt}\underline{z} = A\underline{z} + \underline{q}$ with the nonlinear part given by $\underline{q} := zw \cdot \begin{pmatrix} -\frac{\beta}{M} \\ -\frac{\vartheta}{N} \end{pmatrix}$. Now we can obtain the time derivative of the vector $\underline{x}$ via:

$$\frac{d}{dt}\underline{x} = \Lambda\underline{x} + T^{-1}\underline{q}(\underline{x})$$

Obtaining explicitly:

$$\begin{aligned} \dot{x} &= \quad -\frac{\beta}{M}xy + \frac{c}{d}\frac{1}{k}x^2 \\ \dot{y} &= d \cdot y + \left(\frac{c}{d}\frac{\beta}{M} + \frac{\vartheta}{N}\right)\left(\frac{c}{d}\frac{1}{k}x^2 - \frac{1}{k}xy\right) \end{aligned}$$

To find the transformation $y = h(x)$ along the center manifold, the functional $\mathcal{N}\left(h\left(x\right)\right)$ has to vanish:

$$\mathcal{N}\left(h\left(x\right)\right) = \frac{dh}{dx} \cdot f\left(x, h\left(x\right)\right) - \left(d \cdot h\left(x\right) + g\left(x, h\left(x\right)\right)\right) = 0$$

Center manifold analysis in SISUV

This equation can be solved via polynomial approximation of $h(x)$:

$$h(x) := a_2 \cdot x^2 + a_3 \cdot x^3 + a_4 \cdot x^4 + a_5 \cdot x^5 + \mathcal{O}(x^6)$$

The center manifold was calculated by a 3^{rd} order polynomial:

$$a_2 = -\frac{c \cdot s}{d^2 \cdot k^2}$$

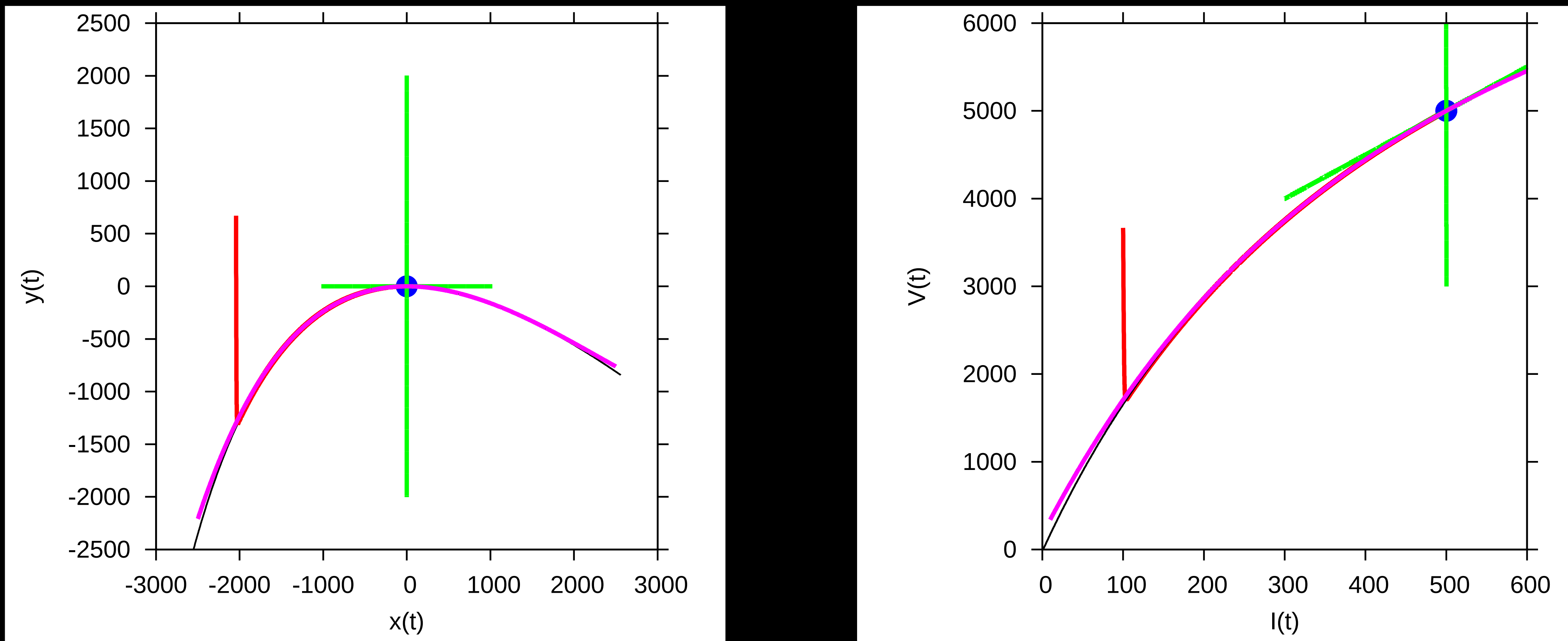
$$a_3 = \frac{1}{d}\left(\frac{2c}{d \cdot k}\frac{\beta}{M} + \frac{s}{k}\right) a_2$$

From the 3^{rd} order polynomial, it is possible to use a general formula to easily get a polynomial of a higer order:

$$a_j = \frac{1}{d}\left((j-1)\frac{\beta}{M}\frac{c}{k \cdot d} + \frac{s}{k}\right) a_{j-1} - \frac{\beta}{M \cdot d}\left(\sum_{\ell=2}^{j-2} \ell \cdot a_j \cdot a_{j-\ell}\right)$$

For j $= 4, 5, ..., \infty$.

Center manifold analysis in SISUV



Analytic seasonal forced SIS

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Considering stable population size $N = S + I$ we can simplify

$$\dot{I} = \frac{\beta}{N}(N - I) I - \alpha I$$

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The seasonal forcing in given by $\beta(t) = \beta_0(1 + \eta \cdot cos(\omega t))$

$$\dot{I} = \frac{\beta(t)}{N}(N - I)I - \alpha I$$

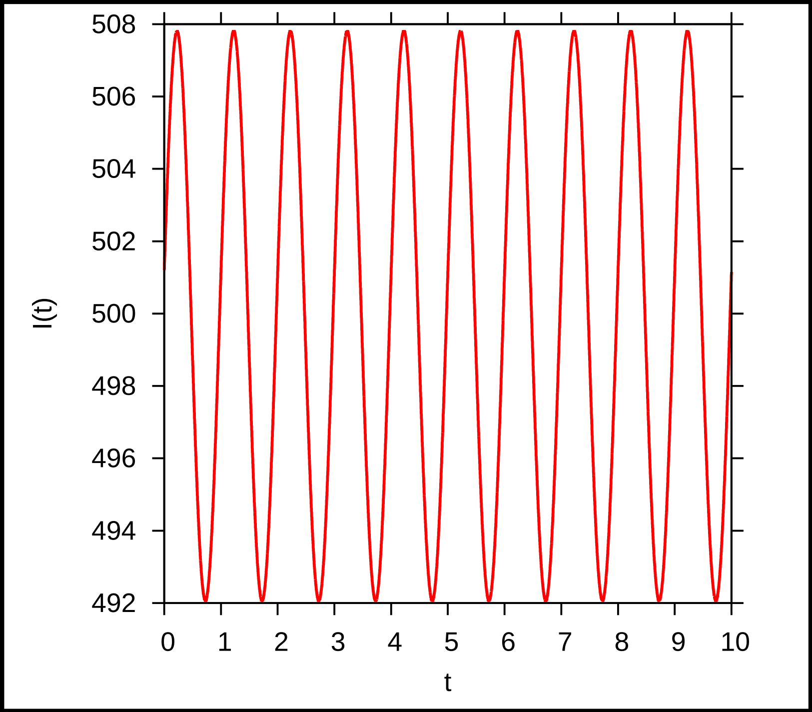
Analytic seasonal forced SIS

In the seasonal forcing we will consider the complex formulation, for now

$$\beta(t) = \beta_0 + \varepsilon\beta_1 e^{i\omega t}$$

Analytic seasonal forced SIS

If we plot the SIS seasonal forced



The $I(t)$ is defined by the stationary state plus some oscillations dependent on the amplitude I_1, *i.e.*

$$I(t) = I_0 + \varepsilon I_1 e^{i\omega t} + \mathcal{O}(\varepsilon^2)$$

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And applying the time derivative to $I(t)$

$$\frac{dI}{dt} = \varepsilon I_1 i\omega e^{i\omega t}$$

Analytic seasonal forced SIS

Substituting in the ODE

$$\frac{d}{dt}I = \frac{\beta(t)}{N}(N-I)\,I - \alpha I$$

$$\varepsilon i\omega I_1 e^{i\omega t} = \frac{1}{N}\left(\beta_0 + \varepsilon\beta_1 e^{i\omega t}\right)\left(N - \left(I_0 + \varepsilon I_1 e^{i\omega t}\right)\right)\left(I_0 + \varepsilon I_1 e^{i\omega t}\right) - \alpha\left(I_0 + \varepsilon I_1 e^{i\omega t}\right)$$

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And separating the terms in respect to order of ε, we get

$$\varepsilon i\omega I_1 e^{i\omega t} = \varepsilon e^{i\omega t}\left(-\alpha I_1 + \frac{1}{N}\left(-\beta_0 I_0 I_1 + I_0\beta_1 N - I_0^2\beta_1 + I_1\beta_0 N - \beta_0 I_0 I_1\right)\right)$$
$$+\varepsilon^0\left(\frac{\beta_0}{N}(N-I_0)I_0 - \alpha I_0\right)$$

Analytic seasonal forced SIS

Substituting in the ODE

$$\frac{d}{dt}I = \frac{\beta(t)}{N}(N-I)I - \alpha I$$

$$\varepsilon i\omega I_1 e^{i\omega t} = \frac{1}{N}\left(\beta_0 + \varepsilon\beta_1 e^{i\omega t}\right)\left(N - \left(I_0 + \varepsilon I_1 e^{i\omega t}\right)\right)\left(I_0 + \varepsilon I_1 e^{i\omega t}\right) - \alpha\left(I_0 + \varepsilon I_1 e^{i\omega t}\right)$$

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The values of order ε^0 have conditions for stationarity, hence $I_0 = I^*$

$$\varepsilon i\omega I_1 e^{i\omega t} = \varepsilon e^{i\omega t}\left(-\alpha I_1 + \frac{1}{N}\left(-\beta_0 I_0 I_1 + I_0\beta_1 N - I_0^2\beta_1 + I_1\beta_0 N - \beta_0 I_0 I_1\right)\right)$$

Analytic seasonal forced SIS

We get the complex amplitude for I_1

$$I_1 = \frac{\frac{\beta_1}{N}\left(N - I_0\right) I_0}{i\omega + \frac{\beta_0}{N} I_0 + \alpha - \frac{\beta_0}{N}\left(N - I_0\right)}$$

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Setting $a := \frac{\beta_0}{N} I_0 + \alpha - \frac{\beta_0}{N}\left(N - I_0\right)$ **and** $c := \frac{\beta_1}{N}\left(N - I_0\right) I_0$ **, we can simplify**

$$I_1 = \frac{c}{a + i\omega}$$

Analytic seasonal forced SIS

We get the complex amplitude for I_1

$$I_1 = \frac{\frac{\beta_1}{N}(N - I_0)\, I_0}{i\omega + \frac{\beta_0}{N} I_0 + \alpha - \frac{\beta_0}{N}(N - I_0)}$$

Setting $a := \frac{\beta_0}{N} I_0 + \alpha - \frac{\beta_0}{N}(N - I_0)$ and $c := \frac{\beta_1}{N}(N - I_0)\, I_0$, we can simplify

$$I_1 = \frac{c}{a + i\omega}$$

And multiplying numerator and denominator by its complex conjugate $a - i\omega$

$$I_1 = \frac{ca}{(a^2 + \omega^2)} + i\left(\frac{-ca}{(a^2 + \omega^2)}\right)$$

Analytic seasonal forced SIS

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And multiplying numerator and denominator by its complex conjugate $a - i\omega$

$$I_1 = \frac{ca}{(a^2 + \omega^2)} + i\left(\frac{-ca}{(a^2 + \omega^2)}\right) := \widetilde{I}_1 + i\widehat{I}_1$$

where the real part $\widetilde{I}_1 := \frac{ca}{(a^2+\omega^2)}$ and the imaginary part $\widehat{I}_1 := \frac{-c\omega}{(a^2+\omega^2)}$ are determined.

Analytic seasonal forced SIS

Hence the complex response of $I(t)$ is given by

$$I(t) = I^* + \varepsilon I_1 e^{i\omega t}$$

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And applying the same calculations for $e^{-i\omega t}$, the second part of the real cos function for I_1 using its complex conjugate $\bar{I}_1 = \widetilde{I}_1 - i\widehat{I}_1$

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Combining the results for $e^{i\omega t}$ and $e^{-i\omega t}$ gives for the real forcing $\beta(t) = \beta_0 + \varepsilon \frac{1}{2} \beta_1 \left(e^{i\omega t} + e^{-i\omega t} \right)$ the real response of the infected

$$I(t) = I^* + \varepsilon \cdot A_I \cdot cos\left(\omega \left(t + \varphi_I \right) \right)$$

Analytic seasonal forced SIS

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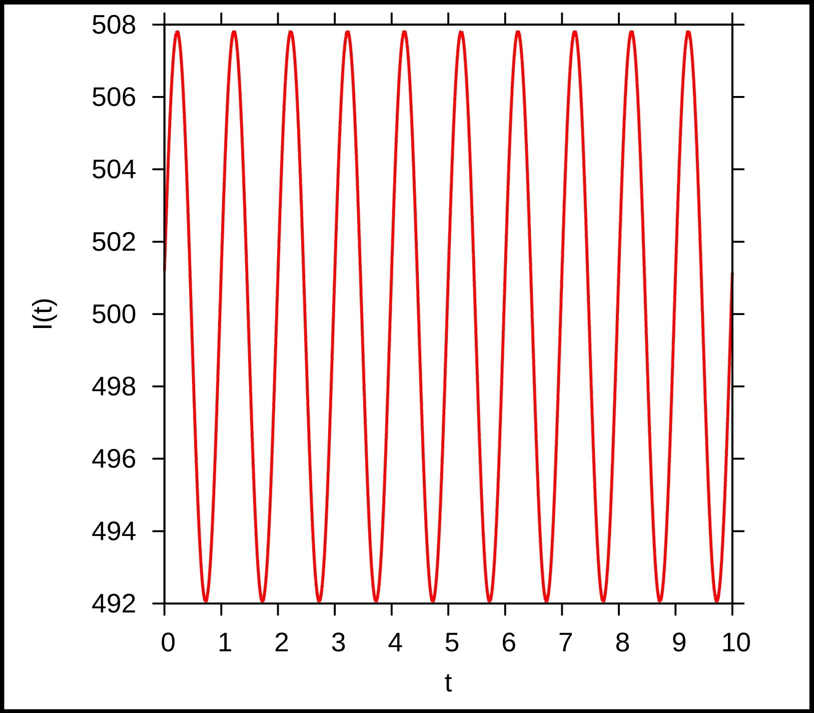
$$I(t) = I^* + \varepsilon \cdot A_I \cdot cos\left(\omega\left(t + \varphi_I\right)\right)$$

with real amplitude A_I and phase φ_I calculated from the complex amplitude

$$A_I = 2\sqrt{\widetilde{I}_1^2 + \widehat{I}_1^2}$$

$$\varphi_I = \frac{1}{\omega} arctan\left(\frac{\widehat{I}_1}{\widetilde{I}_1}\right)$$

Analytic seasonal forced SIS



Parameters: $\alpha = \frac{1}{10}y^{-1}$, $\beta_0 = 2\alpha$ **and** $\eta = 0.1$

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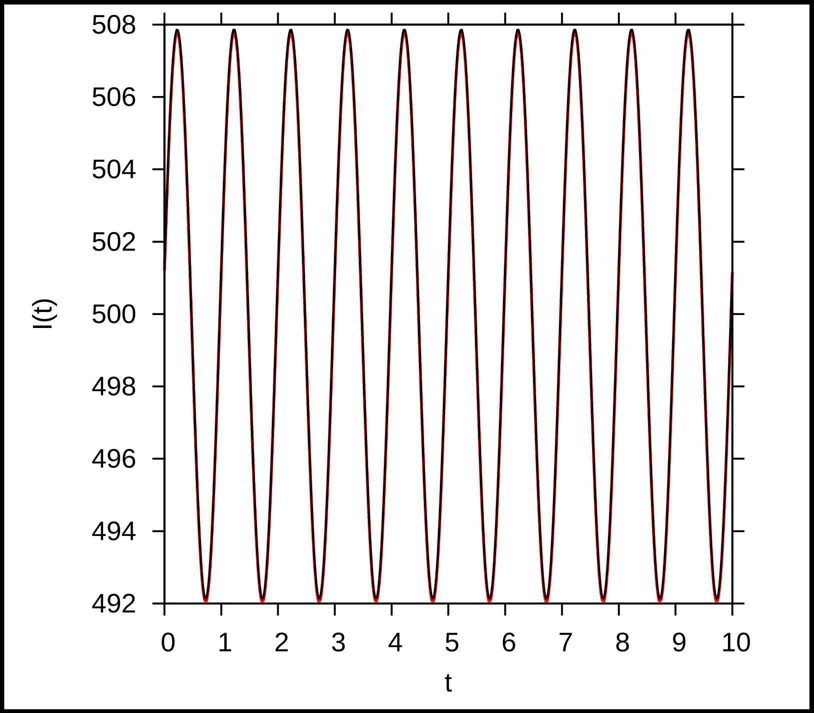

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Analytic seasonally forced SISUV

The next step was to introduce the vector dynamic into the SIS system, getting the SISUV

$$\begin{aligned}
\dot{S} &= \alpha I - \frac{\beta}{M} S V \\
\dot{I} &= \frac{\beta}{M} S V - \alpha I \\
\dot{U} &= \psi - \nu U - \frac{\vartheta}{N} U I \\
\dot{V} &= \frac{\vartheta}{N} U I - \nu V
\end{aligned}$$

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\end{aligned}$$

Considering $N = S(t) + I(t)$ **and** $M = U(t) + V(t)$ **we obtain:**

$$\begin{aligned}
\frac{d}{dt} I &= \frac{\beta}{M}(N - I)V - \alpha I \\
\frac{d}{dt} V &= \frac{\vartheta}{N}(M - V)I - \nu V
\end{aligned}$$

Analytic seasonally forced SISUV

The next step was to introduce the vector dynamic into the SIS system, getting the SISUV

$$\begin{aligned}
\dot{S} &= \alpha I - \frac{\beta}{M} S V \\
\dot{I} &= \frac{\beta}{M} S V - \alpha I \\
\dot{U} &= \psi - \nu U - \frac{\vartheta}{N} U I \\
\dot{V} &= \frac{\vartheta}{N} U I - \nu V
\end{aligned}$$

Considering $N = S(t) + I(t)$ and $M = U(t) + V(t)$ we obtain:

$$\begin{aligned}
\frac{d}{dt} I &= \frac{\beta}{M}(N - I)V - \alpha I \\
\frac{d}{dt} V &= \frac{\vartheta}{N}(M(t) - V)I - \nu V
\end{aligned}$$

With the seasonal forcing given by $M(t) = M_0(1 + \eta \cdot cos(\omega t))$.

Analytic seasonally forced SISUV

The real part of the seasonal forcing is

$$M(t) = M_0 + \varepsilon M_1 e^{i\omega t}$$

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Applying the ansatz we get the general solution for $I(t)$ and $V(t)$

$$I(t) = I_0 + \varepsilon I_1 e^{i\omega t} + \mathcal{O}\left(\varepsilon^2\right)$$

$$V(t) = V_0 + \varepsilon V_1 e^{i\omega t} + \mathcal{O}\left(\varepsilon^2\right)$$

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$$V(t) = V_0 + \varepsilon V_1 e^{i\omega t} + \mathcal{O}\left(\varepsilon^2\right)$$

and the time derivatives

$$\frac{dI}{dt} = \varepsilon I_1 i\omega e^{i\omega t}$$

$$\frac{dV}{dt} = \varepsilon V_1 i\omega e^{i\omega t}$$

Analytic seasonally forced SISUV

Substituting in the ODE for I

$$\frac{d}{dt}I = \frac{\beta}{M}(N - I)V - \alpha I$$

$$\varepsilon I_1 i\omega e^{i\omega t} I = \frac{\beta}{M}\left(N - \left(I_0 + \varepsilon I_1 e^{i\omega t}\right)\right)\left(V_0 + \varepsilon V_1 e^{i\omega t}\right) - \alpha\left(I_0 + \epsilon I_1 e^{i\omega t}\right)$$

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And reorganizing the terms of different orders of ε

$$\varepsilon I_1 i\omega e^{i\omega t} I = \frac{\beta}{M}(N - I_0)V_0 - \alpha I_0 + \varepsilon e^{i\omega t}\left[\frac{\beta}{M}\left(NV_1 - I_0 V_1 - V_0 I_1\right) - \alpha I_1\right]$$

Analytic seasonally forced SISUV

Substituting in the ODE for I

$$\frac{d}{dt}I = \frac{\beta}{M}(N-I)V - \alpha I$$

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And reorganizing the terms of different orders of ε

$$\varepsilon I_1 i\omega e^{i\omega t} I = \frac{\beta}{M}(N-I_0)V_0 - \alpha I_0 + \varepsilon e^{i\omega t}\left[\frac{\beta}{M}\left(NV_1 - I_0V_1 - V_0I_1\right) - \alpha I_1\right]$$

As $I_0 = I^*$ and $V_0 = V^*$ we can say that $\frac{\beta}{M}(N-I_0)V_0 - \alpha I_0 = 0$, so:

$$\varepsilon I_1 i\omega e^{i\omega t} = \varepsilon e^{i\omega t}\left[\frac{\beta}{M}\left(NV_1 - I_0V_1 - V_0I_1\right) - \alpha I_1\right]$$

Analytic seasonally forced SISUV

And finally we get the complex amplitude of I_1 dependent on V_1

$$I_1 = \frac{\frac{\beta}{M}(N - I_0)}{\frac{\beta}{M}V_0 + \alpha + i\omega} V_1$$

Analytic seasonally forced SISUV

And finally we get the complex amplitude of I_1 dependent on V_1

$$I_1 = \frac{\frac{\beta}{M}(N - I_0)}{\frac{\beta}{M}V_0 + \alpha + i\omega}V_1$$

Setting $\frac{\beta}{M}(N - I_0) := c$ and $\frac{\beta}{M}V_0 + \alpha := d$, we get

$$I_1 := \frac{c}{d + i\omega}V_1$$

Analytic seasonally forced SISUV

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$$I_1 := \frac{c}{d + i\omega} V_1$$

And multiplying numerator and denominator by the complex conjugate $\frac{c}{d - i\omega}$, we obtain

$$I_1 = \left(\frac{cd}{d^2 + \omega^2} + i \frac{-c\omega}{d^2 + \omega^2} \right) V_1$$

Analytic seasonally forced SISUV

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$$I_1 := \frac{c}{d + i\omega} V_1$$

And multiplying numerator and denominator by the complex conjugate $\frac{c}{d - i\omega}$, we obtain

$$I_1 = \left(\frac{cd}{d^2 + \omega^2} + i \frac{-c\omega}{d^2 + \omega^2} \right) V_1 =: (a + ib) V_1$$

with $a := \frac{cd}{d^2+\omega^2}$ and $b := \frac{-c\omega}{d^2+\omega^2}$.

Analytic seasonally forced SISUV

Now we apply the same calculations to find the analytic solution of the amplitude for V_1. However, in this case we use the seasonal forcing in $M(t)$

$$\begin{aligned}\frac{d}{dt}V &= \frac{\vartheta}{N}(M(t)-V)I-\nu V\\ \varepsilon V_1 i\omega e^{i\omega t} &= \frac{\vartheta}{N}\left(\left(M_0+\varepsilon M_1 e^{i\omega t}\right)-\left(V_0+\varepsilon V_1 e^{i\omega t}\right)\right)\left(V_0+\varepsilon\left(a+ib\right)V_1 e^{i\omega t}\right)-\\ &\quad -\nu\left(V_0+\varepsilon V_1 e^{i\omega t}\right)\end{aligned}$$

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Now we apply the same calculations to find the analytic solution of the amplitude for V_1. However, in this case we use the seasonal forcing in $M(t)$

$$\begin{aligned}\frac{d}{dt}V &= \frac{\vartheta}{N}(M(t)-V)I-\nu V\\ \varepsilon V_1 i\omega e^{i\omega t} &= \frac{\vartheta}{N}\left(\left(M_0+\varepsilon M_1 e^{i\omega t}\right)-\left(V_0+\varepsilon V_1 e^{i\omega t}\right)\right)\left(V_0+\varepsilon\left(a+ib\right)V_1 e^{i\omega t}\right)-\\ &\quad -\nu\left(V_0+\varepsilon V_1 e^{i\omega t}\right)\end{aligned}$$

Rearranging in orders of ε, we get

$$\begin{aligned}\varepsilon V_1 i\omega e^{i\omega t} &= \frac{\vartheta}{N}\left(M_0-V_0\right)I_0-\nu V_0+\\ &\quad +\varepsilon e^{i\omega t}\left[\frac{\vartheta}{N}M_1 I_0+\left(\frac{\vartheta}{N}\left(M_0(a+ib)-I_0-V_0(a+ib)\right)-\nu\right)V_1\right]\end{aligned}$$

Analytic seasonally forced SISUV

Now we apply the same calculations to find the analytic solution of the amplitude for V_1. However, in this case we use the seasonal forcing in $M(t)$

$$
\begin{aligned}
\frac{d}{dt}V &= \frac{\vartheta}{N}(M(t) - V)I - \nu V \\
\varepsilon V_1 i\omega e^{i\omega t} &= \frac{\vartheta}{N}\left(\left(M_0 + \varepsilon M_1 e^{i\omega t}\right) - \left(V_0 + \varepsilon V_1 e^{i\omega t}\right)\right)\left(V_0 + \varepsilon\left(a + ib\right)V_1 e^{i\omega t}\right) - \\
&\quad -\nu\left(V_0 + \varepsilon V_1 e^{i\omega t}\right)
\end{aligned}
$$

Rearranging in orders of ε, we get

$$
\begin{aligned}
\varepsilon V_1 i\omega e^{i\omega t} &= \frac{\vartheta}{N}\left(M_0 - V_0\right)I_0 - \nu V_0 + \\
&\quad +\varepsilon e^{i\omega t}\left[\frac{\vartheta}{N}M_1 I_0 + \left(\frac{\vartheta}{N}\left(M_0(a+ib) - I_0 - V_0(a+ib)\right) - \nu\right)V_1\right]
\end{aligned}
$$

Once again, we can forget about the terms of the order ε^0

$$
V_1 i\omega = \frac{\vartheta}{N}M_1 I_0 + \left(\frac{\vartheta}{N}\left(M_0(a+ib) - I_0 - V_0(a+ib)\right) - \nu\right)V_1
$$

Analytic seasonally forced SISUV

Obtaining specifically

$$V_1 = \frac{\frac{\vartheta}{N} M_1 I_0}{\frac{\vartheta}{N}\left(I_0 + a(V_0 - M_0)\right) + \nu + i\left(\frac{\vartheta}{N} b(V_0 - M_0) + \omega\right)}$$

Analytic seasonally forced SISUV

Obtaining specifically

$$V_1 = \frac{\frac{\vartheta}{N}M_1 I_0}{\frac{\vartheta}{N}\left(I_0 + a(V_0 - M_0)\right) + \nu + i\left(\frac{\vartheta}{N}b(V_0 - M_0) + \omega\right)}$$

Setting $u := \frac{\vartheta}{N}\left(I_0 + a(V_0 - M_0)\right) + \nu$, $v := \frac{\vartheta}{N}b(V_0 - M_0) + \omega$ **and** $w := \frac{\vartheta}{N}M_1 I_0$.

$$V_1 =: \frac{w}{u + iv}$$

Analytic seasonally forced SISUV

Obtaining specifically

$$V_1 = \frac{\frac{\vartheta}{N} M_1 I_0}{\frac{\vartheta}{N}\left(I_0 + a(V_0 - M_0)\right) + \nu + i\left(\frac{\vartheta}{N} b(V_0 - M_0) + \omega\right)}$$

Setting $u := \frac{\vartheta}{N}\left(I_0 + a(V_0 - M_0)\right) + \nu$, $v := \frac{\vartheta}{N} b(V_0 - M_0) + \omega$ and $w := \frac{\vartheta}{N} M_1 I_0$.

$$V_1 =: \frac{w}{u + iv}$$

Multiplying both terms by the complex conjugate, we get

$$V_1 = \frac{wu}{u^2 + v^2} + i\frac{-wv}{u^2 + v^2} =: \widetilde{V}_1 + i\widehat{V}_1$$

being $\widetilde{V}_1 := \frac{wu}{u^2+v^2}$ and $\widehat{V}_1 := \frac{-wv}{u^2+v^2}$, obtaining the complex amplitude for V_1 .

Analytic seasonally forced SISUV

And substituting in the analytic expression for complex amplitude for $\boldsymbol{I_1}$, we obtain

$$\boldsymbol{I_1 = (a + ib) \cdot (\widetilde{V}_1 + i\widehat{V}_1)}$$

Analytic seasonally forced SISUV

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Analytic seasonally forced SISUV

And substituting in the analytic expression for complex amplitude for $\boldsymbol{I_1}$, we obtain

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$$\boldsymbol{I_1 \; = \; a\widetilde{V}_1 - b\widehat{V}_1 + i(a\widehat{V}_1 + b\widetilde{V}_1)}$$

And setting $\boldsymbol{\widetilde{I}_1 := a\widetilde{V}_1 - b\widehat{V}_1}$ and $\boldsymbol{\widehat{I}_1 := a\widehat{V}_1 + b\widetilde{V}_1}$, we can simplify

$$\boldsymbol{I_1 =: \widetilde{I}_1 + i\widehat{I}_1}$$

Analytic seasonally forced SISUV

Obtaining the complex response for both $I(t)$ and $V(t)$ with the real and complex parts of each one and doing a similar analysis for the second part of the real cos function, as was done for SIS, we can obtain the real response for the system:

Analytic seasonally forced SISUV

Obtaining the complex response for both $I(t)$ and $V(t)$ with the real and complex parts of each one and doing a similar analysis for the second part of the real *cos* function, as was done for SIS, we can obtain the real response for the system:

$$I(t) = I^* + \varepsilon \cdot A_I \cdot cos\left(\omega\left(t + \varphi_I\right)\right)$$

$$V(t) = V^* + \varepsilon \cdot A_V \cdot cos\left(\omega\left(t + \varphi_V\right)\right)$$

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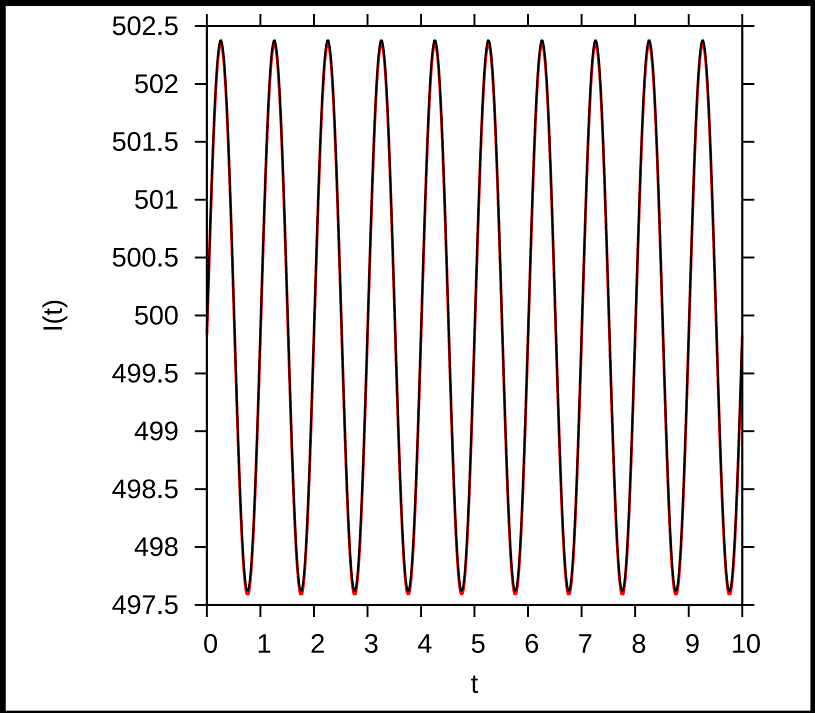
$$V(t) = V^* + \varepsilon \cdot A_V \cdot cos\left(\omega\left(t + \varphi_V\right)\right)$$

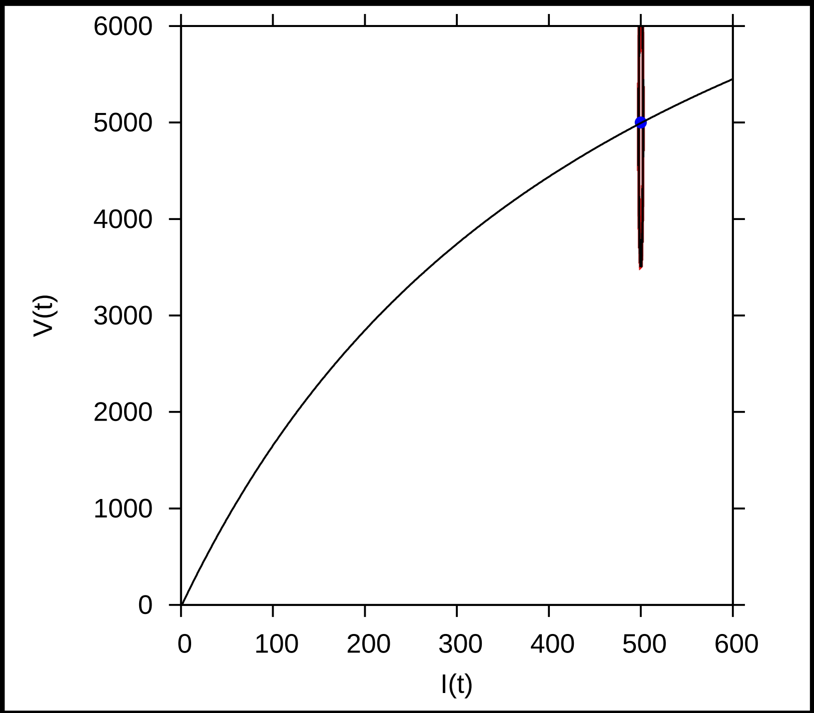
with the amplitude and phase for both variables given by

$$A_I = 2\sqrt{\widetilde{I}_1^2 + \widehat{I}_1^2} \qquad \varphi_I = \frac{1}{\omega} arctan\left(\frac{\widehat{I}_1}{\widetilde{I}_1}\right)$$

$$A_V = 2\sqrt{\widetilde{V}_1^2 + \widehat{V}_1^2} \qquad \varphi_V = \frac{1}{\omega} arctan\left(\frac{\widehat{V}_1}{\widetilde{V}_1}\right)$$

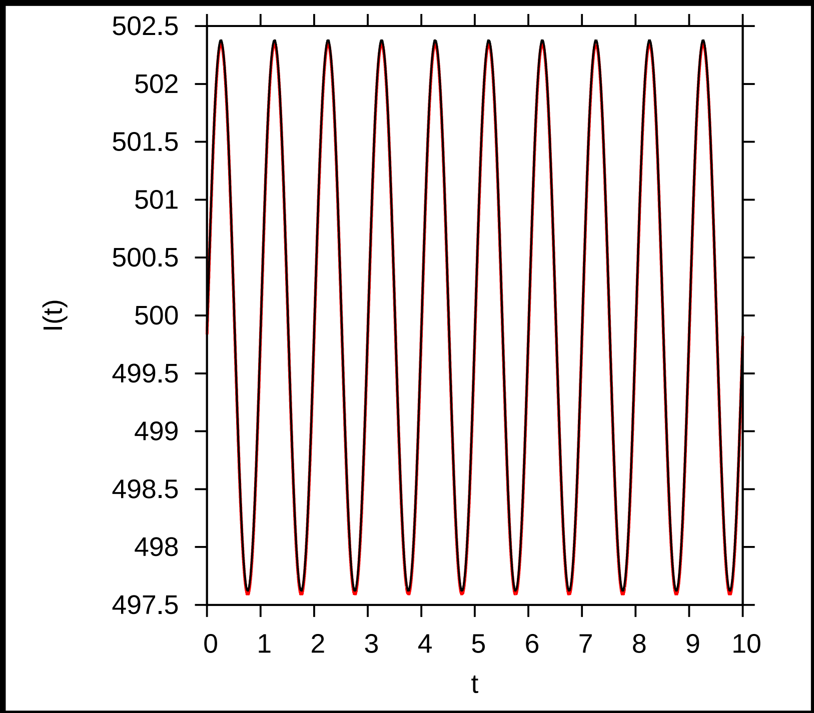
Analytic seasonally forced SISUV

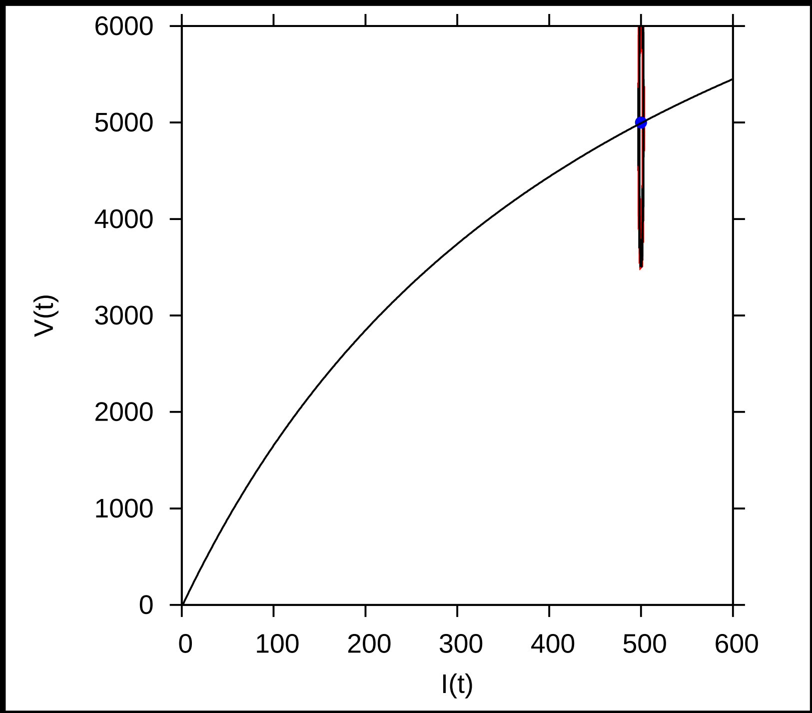




Parameters: $\alpha = \frac{1}{10} y^{-1}$, $\beta = 2\alpha$, $\nu = \frac{365}{10} d^{-1}$, $\vartheta = 2\nu$ **and** $\eta = 0.3$.

Analytic seasonally forced SISUV





Parameters: $\alpha = \frac{1}{10} y^{-1}$, $\beta = 2\alpha$, $\nu = \frac{365}{10} d^{-1}$, $\vartheta = 2\nu$ and $\eta = 0.3$.

The amplitude in $V(t)$ caused by the seasonality is not reflected in $I(t)$ dynamics.

So, for modelling proposes, the vector dynamics is not important for the system.

The Full SIRUV and comparison with SIR model

After have analysed the simplest models we made similar calculation for more complicated SIR and SIRUV model, comparing the expressions from each other.

The Full SIRUV and comparison with SIR model

After have analysed the simplest models we made similar calculation for more complicated SIR and SIRUV model, comparing the expressions from each other.

Considering a closed population for humans $N(t) = S(t)+I(t)+R(t)$ we can simplify the SIR model into a two dimentional system:

$$\frac{d}{dt}I = \frac{\beta(t)}{N} \cdot (N - I - R) \cdot I - (\mu + \gamma) \cdot I$$

$$\frac{d}{dt}R = \gamma \cdot I - \mu \cdot R$$

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$$\frac{d}{dt}R = \gamma \cdot I - \mu \cdot R$$

And also for vectors $M(t) = U(t) + V(t)$ for SIRUV model

$$\frac{d}{dt}I = \frac{\beta}{M_0} \cdot (N - I - R) \cdot V - (\gamma + \mu) \cdot I$$

$$\frac{d}{dt}R = \gamma \cdot I - \mu \cdot R$$

$$\frac{d}{dt}V = \frac{\vartheta}{N} \cdot (M(t) - V) \cdot I - \nu \cdot V \quad .$$

The Full SIRUV and comparison with SIR model

The endemic stationary states of the SIR are given by

$$
\begin{aligned}
I^* &= \frac{\mu}{(\gamma+\mu)} \cdot \left(1 - \frac{\gamma+\mu}{\beta}\right) \cdot N \\
R^*(I^*) &= \frac{\gamma}{\mu} I^*
\end{aligned}
$$

The Full SIRUV and comparison with SIR model

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Whereas the stationary states for SIRUV system are given by

$$\begin{aligned} I^* &= \frac{\mu}{(\gamma+\mu) + \frac{\gamma+\mu}{\beta} \cdot \mu} \cdot \left(1 - \frac{\gamma+\mu}{\frac{\vartheta}{\nu}\beta}\right) \cdot N \\ R^*(I^*) &= \frac{\gamma}{\mu} I^* \\ V^*(I^*) &= \frac{\frac{\vartheta}{N} I^*}{\nu + \frac{\vartheta}{N} I^*} \cdot M \quad . \end{aligned}$$

The Full SIRUV and comparison with SIR model

The endemic stationary states of the SIR are given by

$$\begin{aligned} \boldsymbol{I}^* &= \frac{\boldsymbol{\mu}}{(\boldsymbol{\gamma}+\boldsymbol{\mu})} \cdot \left(1 - \frac{\boldsymbol{\gamma}+\boldsymbol{\mu}}{\boldsymbol{\beta}}\right) \cdot \boldsymbol{N} \\ \boldsymbol{R}^*(\boldsymbol{I}^*) &= \frac{\boldsymbol{\gamma}}{\boldsymbol{\mu}} \boldsymbol{I}^* \end{aligned}$$

Whereas the stationary states for SIRUV system are given by

$$\begin{aligned} \boldsymbol{I}^* &= \frac{\boldsymbol{\mu}}{(\boldsymbol{\gamma}+\boldsymbol{\mu}) + \frac{\boldsymbol{\gamma}+\boldsymbol{\mu}}{\boldsymbol{\beta}} \cdot \boldsymbol{\mu}} \cdot \left(1 - \frac{\boldsymbol{\gamma}+\boldsymbol{\mu}}{\frac{\boldsymbol{\vartheta}}{\boldsymbol{\nu}}\boldsymbol{\beta}}\right) \cdot \boldsymbol{N} \\ \boldsymbol{R}^*(\boldsymbol{I}^*) &= \frac{\boldsymbol{\gamma}}{\boldsymbol{\mu}} \boldsymbol{I}^* \\ \boldsymbol{V}^*(\boldsymbol{I}^*) &= \frac{\frac{\boldsymbol{\vartheta}}{\boldsymbol{N}}\boldsymbol{I}^*}{\boldsymbol{\nu} + \frac{\boldsymbol{\vartheta}}{\boldsymbol{N}}\boldsymbol{I}^*} \cdot \boldsymbol{M} \quad . \end{aligned}$$

Essentially we can obtain the same stationary state in both models, by replacing the $\boldsymbol{\beta}$ by $\frac{\boldsymbol{\vartheta}}{\boldsymbol{\nu}}\boldsymbol{\beta}$ in SIR.

The Full SIRUV and comparison with SIR model

The seasonal forcing was included via infection rate $\beta(t)$ for SIR and via total number of mosquitos $M(t)$ for SIRUV.

The Full SIRUV and comparison with SIR model

The seasonal forcing was included via infection rate $\beta(t)$ for SIR and via total number of mosquitos $M(t)$ for SIRUV.

For small seasonal forcing η, hence small ε we expect also small oscillations of the state variables, hence

$$I(t) = I_0 + \varepsilon I_1 e^{i\omega t} + \mathcal{O}\left(\varepsilon^2\right)$$

$$R(t) = R_0 + \varepsilon R_1 e^{i\omega t} + \mathcal{O}\left(\varepsilon^2\right)$$

$$V(t) = V_0 + \varepsilon V_1 e^{i\omega t} + \mathcal{O}\left(\varepsilon^2\right)$$

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$$V(t) = V_0 + \varepsilon V_1 e^{i\omega t} + \mathcal{O}\left(\varepsilon^2\right)$$

and for the time derivatives

$$\frac{dI}{dt} = \varepsilon I_1 i\omega e^{i\omega t}$$

$$\frac{dR}{dt} = \varepsilon R_1 i\omega e^{i\omega t}$$

$$\frac{dV}{dt} = \varepsilon V_1 i\omega e^{i\omega t} \quad .$$

The Full SIRUV and comparison with SIR model

The R is the same for both models, so

$$\frac{d}{dt}R = \gamma I - \mu R$$

$$\varepsilon R_1 i\omega e^{i\omega t} = \gamma\left(I_0 + \varepsilon I_1 e^{i\omega t}\right) - \mu\left(R_0 + \varepsilon R_1 e^{i\omega t}\right)$$

The Full SIRUV and comparison with SIR model

The R is the same for both models, so

$$\begin{aligned} \frac{d}{dt}R &= \gamma I - \mu R \\ \varepsilon R_1 i\omega e^{i\omega t} &= \gamma\left(I_0 + \varepsilon I_1 e^{i\omega t}\right) - \mu\left(R_0 + \varepsilon R_1 e^{i\omega t}\right) \\ &= \gamma I_0 - \mu R_0 + \varepsilon e^{i\omega t}\left(\gamma I_1 - \mu R_1\right) \end{aligned}$$

The Full SIRUV and comparison with SIR model

The R is the same for both models, so

$$\frac{d}{dt}R = \gamma I - \mu R$$

$$\varepsilon R_1 i\omega e^{i\omega t} = \gamma \left(I_0 + \varepsilon I_1 e^{i\omega t}\right) - \mu \left(R_0 + \varepsilon R_1 e^{i\omega t}\right)$$

$$= \gamma I_0 - \mu R_0 + \varepsilon e^{i\omega t} \left(\gamma I_1 - \mu R_1\right)$$

To zeroth order ε^0 we obtain again the condition for stationarity

$$R_1 = \frac{\gamma}{\mu + i\omega} I_1$$

The Full SIRUV and comparison with SIR model

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&= \gamma I_0 - \mu R_0 + \varepsilon e^{i\omega t}\left(\gamma I_1 - \mu R_1\right)
\end{aligned}$$

To zeroth order ε^0 we obtain again the condition for stationarity

$$R_1 = \frac{\gamma}{\mu + i\omega} I_1$$

Finally, multiplying numerator and denomicator both by its complex conjugate we obtain

$$R_1 = \left(\frac{\gamma\mu}{\mu^2+\omega^2} + i\frac{-\gamma\omega}{\mu^2+\omega^2}\right)\cdot I_1$$

The Full SIRUV and comparison with SIR model

The $\boldsymbol{R}$ is the same for both models, so

$$\begin{aligned}
\frac{d}{dt}R &= \gamma I - \mu R \\
\varepsilon R_1 i\omega e^{i\omega t} &= \gamma\left(I_0 + \varepsilon I_1 e^{i\omega t}\right) - \mu\left(R_0 + \varepsilon R_1 e^{i\omega t}\right) \\
&= \gamma I_0 - \mu R_0 + \varepsilon e^{i\omega t}\left(\gamma I_1 - \mu R_1\right)
\end{aligned}$$

To zeroth order ε^0 we obtain again the condition for stationarity

$$R_1 = \frac{\gamma}{\mu + i\omega} I_1$$

Finally, multiplying numerator and denomicator both by its complex conjugate we obtain

$$R_1 = \left(\frac{\gamma\mu}{\mu^2+\omega^2} + i\frac{-\gamma\omega}{\mu^2+\omega^2}\right)\cdot I_1 =: (a+ib)I_1$$

with $a := \frac{\gamma\mu}{\mu^2+\omega^2} b := \frac{-\gamma\omega}{\mu^2+\omega^2}$.

The Full SIRUV and comparison with SIR model

Now we are going to analyse the I for both models.

$$\frac{dI}{dt} = \frac{\beta(t)}{N} \cdot (N - I - R) \cdot I - (\mu + \gamma) \cdot I \qquad \frac{d}{dt}I = \frac{\beta}{M} \cdot (N - I - R) \cdot V - (\gamma + \mu) \cdot I$$

The Full SIRUV and comparison with SIR model

Now we are going to analyse the I for both models.

$$\frac{d}{dt}I = \frac{\beta(t)}{N} \cdot (N - I - R) \cdot I - (\mu + \gamma) \cdot I \qquad \frac{d}{dt}I = \frac{\beta}{M} \cdot (N - I - R) \cdot V - (\gamma + \mu) \cdot I$$

$$I_1 = \frac{\frac{\beta_1}{N}(N-I_0-R_0)}{\frac{\beta_0}{N}(1+a)I_0 - \frac{\beta_0}{N}(N-I_0-R_0) + (\gamma+\mu) + i\left(\omega + \frac{\beta_0}{N} b I_0\right)} I_0 \qquad I_1 = \frac{\frac{\beta}{M_0}(N-I_0-R_0)}{\frac{\beta}{M_0}(1+a)V_0 + (\gamma+\mu) + i\left(\omega + \frac{\beta}{M_0} b V_0\right)} V_1$$

The Full SIRUV and comparison with SIR model

Now we are going to analyse the I for both models.

$$\frac{d}{dt}I = \frac{\beta(t)}{N} \cdot (N - I - R) \cdot I - (\mu + \gamma) \cdot I \qquad \frac{d}{dt}I = \frac{\beta}{M} \cdot (N - I - R) \cdot V - (\gamma + \mu) \cdot I$$

$$I_1 = \frac{\frac{\beta_1}{N}(N - I_0 - R_0)}{\frac{\beta_0}{N}(1+a)I_0 - \frac{\beta_0}{N}(N - I_0 - R_0) + (\gamma+\mu) + i\left(\omega + \frac{\beta_0}{N}bI_0\right)} I_0 \qquad I_1 = \frac{\frac{\beta}{M_0}(N - I_0 - R_0)}{\frac{\beta}{M_0}(1+a)V_0 + (\gamma+\mu) + i\left(\omega + \frac{\beta}{M_0}bV_0\right)} V_1$$

Putting both in a $I_1 := \frac{f}{c+id}$ form, we have the specific values of each abreviation.

$$c_{IR} = \frac{\beta_0}{N}(1+a)\,I_0 - \frac{\beta_0}{N}(N - I_0 - R_0) + (\gamma + \mu) \qquad c_{IRV} = \frac{\beta}{M_0}(1+a)V_0 + (\gamma + \mu)$$

$$d_{IR} = \omega + \frac{\beta_0}{N} b I_0 \qquad d_{IRV} = \omega + \frac{\beta}{M_0} b V_0$$

$$f_{IR} = \frac{\beta_1}{N}(N - I_0 - R_0)\,I_0 \qquad f_{IRV} = \frac{\beta}{M_0}(N - I_0 - R_0)$$

The Full SIRUV and comparison with SIR model

Now we are going to analyse the I for both models.

$$\frac{d}{dt}I = \frac{\beta(t)}{N} \cdot (N - I - R) \cdot I - (\mu + \gamma) \cdot I \qquad \frac{d}{dt}I = \frac{\beta}{M} \cdot (N - I - R) \cdot V - (\gamma + \mu) \cdot I$$

$$I_1 = \frac{\frac{\beta_1}{N}(N - I_0 - R_0)}{\frac{\beta_0}{N}(1+a)I_0 - \frac{\beta_0}{N}(N - I_0 - R_0) + (\gamma+\mu) + i\left(\omega + \frac{\beta_0}{N}bI_0\right)} I_0 \qquad I_1 = \frac{\frac{\beta}{M_0}(N - I_0 - R_0)}{\frac{\beta}{M_0}(1+a)V_0 + (\gamma+\mu) + i\left(\omega + \frac{\beta}{M_0}bV_0\right)} V_1$$

Putting both in a $I_1 := \frac{f}{c+id}$ form, we have the specific values of each abreviation.

$$c_{IR} = \frac{\beta_0}{N}(1 + a)\, I_0 - \frac{\beta_0}{N}(N - I_0 - R_0) + (\gamma + \mu) \qquad c_{IRV} = \frac{\beta}{M_0}(1 + a)V_0 + (\gamma + \mu)$$

$$d_{IR} = \omega + \frac{\beta_0}{N} b I_0 \qquad d_{IRV} = \omega + \frac{\beta}{M_0} b V_0$$

$$f_{IR} = \frac{\beta_1}{N}(N - I_0 - R_0)\, I_0 \qquad f_{IRV} = \frac{\beta}{M_0}(N - I_0 - R_0)$$

Essencially, the only difference between the models is given by $-\frac{\beta_0}{N}(N - I_0 - R_0)$ in the SIR.

The Full SIRUV and comparison with SIR model

Multiplying both terms of I_1 by the complex conjugate $c - id$ we obtain the amplitude of I_1 for models.

$$I_1 := \frac{f_{IR}}{c_{IR} + i d_{IR}} \qquad I_1 := \frac{f_{IRV}}{c_{IRV} + i d_{IRV}} V_1$$

$$I_1 = \left(\left(\frac{f_{IR} c_{IR}}{c_{IR}^2 + d_{IR}^2} \right) + i \left(\frac{-f_{IR} d_{IR}}{c_{IR}^2 + d_{IR}^2} \right) \right) \qquad I_1 = \left(\left(\frac{f_{IRV} c_{IRV}}{c_{IRV}^2 + d_{IRV}^2} \right) + i \left(\frac{-f_{IRV} d_{IRV}}{c_{IRV}^2 + d_{IRV}^2} \right) \right) \cdot V_1$$

The Full SIRUV and comparison with SIR model

Multiplying both terms of I_1 by the complex conjugate $c-id$ we obtain the amplitude of I_1 for models.

$$I_1 := \frac{f_{IR}}{c_{IR}+id_{IR}} \qquad I_1 := \frac{f_{IRV}}{c_{IRV}+id_{IRV}}V_1$$

$$I_1 = \left(\left(\frac{f_{IR}c_{IR}}{c_{IR}^2+d_{IR}^2}\right) + i\left(\frac{-f_{IR}d_{IR}}{c_{IR}^2+d_{IR}^2}\right)\right) \qquad I_1 = \left(\left(\frac{f_{IRV}c_{IRV}}{c_{IRV}^2+d_{IRV}^2}\right) + i\left(\frac{-f_{IRV}d_{IRV}}{c_{IRV}^2+d_{IRV}^2}\right)\right)\cdot V_1$$

$$I_1 := \widetilde{I}_1 + i\widehat{I}_1 \qquad I_1 := (x+iy)V_1$$

Being $\widetilde{I}_1 \triangleq x := \frac{fc}{c^2+d^2}$ and $\widehat{I}_1 \triangleq y := \frac{-fd}{c^2+d^2}$.

The Full SIRUV and comparison with SIR model

Finally, from the ODE for the infected mosquitos

$$\frac{d}{dt}V = \frac{\vartheta}{N}(M(t) - V)I - \nu V$$

The Full SIRUV and comparison with SIR model

Finally, from the ODE for the infected mosquitos

$$\frac{d}{dt}V = \frac{\vartheta}{N}(M(t) - V)I - \nu V$$

$$V_1 = \frac{\frac{\vartheta}{N}M_1 I_0}{\frac{\vartheta}{N}(I_0 - (M_0 - V_0)x) + \nu + i\omega - \frac{\vartheta}{N}(M_0 - V_0)y}$$

The Full SIRUV and comparison with SIR model

Finally, from the ODE for the infected mosquitos

$$
\begin{aligned}
\frac{d}{dt}V &= \frac{\vartheta}{N}(M(t) - V)I - \nu V \\
V_1 &= \frac{\frac{\vartheta}{N}M_1 I_0}{\frac{\vartheta}{N}(I_0 - (M_0 - V_0)x) + \nu + i\omega - \frac{\vartheta}{N}(M_0 - V_0)y} \\
V_1 &:= \frac{k}{g + ih}
\end{aligned}
$$

with the coefficients $g := \frac{\vartheta}{N}(I_0 - (M_0 - V_0)x) + \nu$, $h := \omega - \frac{\vartheta}{N}(M_0 - V_0)y$ **and** $k := \frac{\vartheta}{N}M_1 I_0$.

The Full SIRUV and comparison with SIR model

Finally, from the ODE for the infected mosquitos

$$
\begin{aligned}
\frac{d}{dt}V &= \frac{\vartheta}{N}(M(t)-V)I - \nu V \\
V_1 &= \frac{\frac{\vartheta}{N}M_1 I_0}{\frac{\vartheta}{N}(I_0-(M_0-V_0)x)+\nu+i\omega-\frac{\vartheta}{N}(M_0-V_0)y} \\
V_1 &:= \frac{k}{g+ih}
\end{aligned}
$$

with the coefficients $g := \frac{\vartheta}{N}(I_0-(M_0-V_0)x)+\nu,\quad h := \omega-\frac{\vartheta}{N}(M_0-V_0)y$ **and** $k := \frac{\vartheta}{N}M_1 I_0$**.**

Multiplying both, numerator and denominator by the complex conjugate ($g-ih$**)**

$$
V_1 = \left(\frac{kg}{g^2+h^2}\right) + i\left(\frac{-kh}{g^2+h^2}\right)
$$

The Full SIRUV and comparison with SIR model

Finally, from the ODE for the infected mosquitos

$$\frac{d}{dt}V = \frac{\vartheta}{N}(M(t) - V)I - \nu V$$

$$V_1 = \frac{\frac{\vartheta}{N}M_1 I_0}{\frac{\vartheta}{N}(I_0 - (M_0 - V_0)x) + \nu + i\omega - \frac{\vartheta}{N}(M_0 - V_0)y}$$

$$V_1 := \frac{k}{g + ih}$$

with the coefficients $g := \frac{\vartheta}{N}(I_0 - (M_0 - V_0)x) + \nu$, $h := \omega - \frac{\vartheta}{N}(M_0 - V_0)y$ and $k := \frac{\vartheta}{N}M_1 I_0$.

Multiplying both, numerator and denominator by the complex conjugate ($g - ih$)

$$V_1 = \left(\frac{kg}{g^2 + h^2}\right) + i\left(\frac{-kh}{g^2 + h^2}\right) := \widetilde{V}_1 + i\widehat{V}_1$$

where the real part is $\widetilde{V}_1 := \frac{kg}{g^2+h^2}$ and the imaginary is $\widehat{V}_1 := \frac{-kh}{g^2+h^2}$.

The Full SIRUV and comparison with SIR model

We can now substitute the complex amplitude of $\boldsymbol{V_1}$ in the amplitude of $\boldsymbol{I_1}$

$$\boldsymbol{I_1 \;=\; (x + iy) \cdot (\widetilde{V}_1 + i\widehat{V}_1)}$$

The Full SIRUV and comparison with SIR model

We can now substitute the complex amplitude of $\boldsymbol{V_1}$ in the amplitude of $\boldsymbol{I_1}$

$$\boldsymbol{I_1 = (x + iy) \cdot (\widetilde{V}_1 + i\widehat{V}_1)}$$

$$\boldsymbol{I_1 = x\widetilde{V}_1 - y\widehat{V}_1 + i\left(x\widehat{V}_1 + y\widetilde{V}_1\right)}$$

The Full SIRUV and comparison with SIR model

We can now substitute the complex amplitude of V_1 in the amplitude of I_1

$$I_1 = (x + iy) \cdot (\widetilde{V}_1 + i\widehat{V}_1)$$

$$I_1 = x\widetilde{V}_1 - y\widehat{V}_1 + i\left(x\widehat{V}_1 + y\widetilde{V}_1\right)$$

$$I_1 := \widetilde{I}_1 + i\widehat{I}_1$$

where the real part is $\widetilde{I}_1 := x\widetilde{V}_1 - y\widehat{V}_1$ and the imaginary is $\widehat{V}_1 := x\widehat{V}_1 + y\widetilde{V}_1$.

The Full SIRUV and comparison with SIR model

We can now substitute the complex amplitude of V_1 in the amplitude of I_1

$$I_1 \;=\; (x+iy)\cdot(\widetilde{V}_1+i\widehat{V}_1)$$

$$I_1 \;=\; x\widetilde{V}_1 - y\widehat{V}_1 + i\left(x\widehat{V}_1 + y\widetilde{V}_1\right)$$

$$I_1 \;:=\; \widetilde{I}_1 + i\widehat{I}_1$$

where the real part is $\widetilde{I}_1 := x\widetilde{V}_1 - y\widehat{V}_1$ and the imaginary is $\widehat{V}_1 := x\widehat{V}_1 + y\widetilde{V}_1$.

And then we can obtain the complex amplitude for R_1, by substituting the value of the complex amplitude of I_1

$$R_1 \;=\; (a+ib)\cdot(\widetilde{I}_1+i\widehat{I}_1)$$

The Full SIRUV and comparison with SIR model

We can now substitute the complex amplitude of V_1 in the amplitude of I_1

$$I_1 \;=\; (x+iy)\cdot(\widetilde{V}_1 + i\widehat{V}_1)$$

$$I_1 \;=\; x\widetilde{V}_1 - y\widehat{V}_1 + i\left(x\widehat{V}_1 + y\widetilde{V}_1\right)$$

$$I_1 \;:=\; \widetilde{I}_1 + i\widehat{I}_1$$

where the real part is $\widetilde{I}_1 := x\widetilde{V}_1 - y\widehat{V}_1$ and the imaginary is $\widehat{V}_1 := x\widehat{V}_1 + y\widetilde{V}_1$.

And then we can obtain the complex amplitude for R_1, by substituting the value of the complex amplitude of I_1

$$R_1 \;=\; (a+ib)\cdot(\widetilde{I}_1 + i\widehat{I}_1)$$

$$R_1 \;=\; a\widetilde{I}_1 - b\widehat{I}_1 + i\left(a\widehat{I}_1 + b\widetilde{I}_1\right)$$

The Full SIRUV and comparison with SIR model

We can now substitute the complex amplitude of V_1 in the amplitude of I_1

$$I_1 = (x + iy) \cdot (\widetilde{V}_1 + i\widehat{V}_1)$$

$$I_1 = x\widetilde{V}_1 - y\widehat{V}_1 + i\left(x\widehat{V}_1 + y\widetilde{V}_1\right)$$

$$I_1 := \widetilde{I}_1 + i\widehat{I}_1$$

where the real part is $\widetilde{I}_1 := x\widetilde{V}_1 - y\widehat{V}_1$ and the imaginary is $\widehat{V}_1 := x\widehat{V}_1 + y\widetilde{V}_1$.

And then we can obtain the complex amplitude for R_1, by substituting the value of the complex amplitude of I_1

$$R_1 = (a + ib) \cdot (\widetilde{I}_1 + i\widehat{I}_1)$$

$$R_1 = a\widetilde{I}_1 - b\widehat{I}_1 + i\left(a\widehat{I}_1 + b\widetilde{I}_1\right)$$

$$R_1 := \widetilde{R}_1 + i\widehat{R}_1$$

where the real part is $\widetilde{R}_1 := a\widetilde{I}_1 - b\widehat{I}_1$ and the imaginary part is $\widehat{R}_1 := a\widehat{I}_1 + b\widetilde{I}_1$.

The Full SIRUV and comparison with SIR model

So we have already obtained the first part of the real $\boldsymbol{cos}$ **function for the three variables**

$$\boldsymbol{I(t) \;=\; I^* + \varepsilon \cdot (\tilde{I}_1 + i\hat{I}_1)e^{i\omega t}}$$

$$\boldsymbol{R(t) \;=\; R^* + \varepsilon \cdot (\tilde{R}_1 + i\hat{R}_1)e^{i\omega t}}$$

$$\boldsymbol{V(t) \;=\; V^* + \varepsilon \cdot (\tilde{V}_1 + i\hat{V}_1)e^{i\omega t}}$$

The Full SIRUV and comparison with SIR model

So we have already obtained the first part of the real *cos* function for the three variables

$$I(t) = I^* + \varepsilon \cdot (\tilde{I}_1 + i\hat{I}_1)e^{i\omega t}$$

$$R(t) = R^* + \varepsilon \cdot (\tilde{R}_1 + i\hat{R}_1)e^{i\omega t}$$

$$V(t) = V^* + \varepsilon \cdot (\tilde{V}_1 + i\hat{V}_1)e^{i\omega t}$$

And for the second part of the real *cos* function

$$I(t) = I^* + \varepsilon \cdot (\tilde{I}_1 - i\hat{I}_1)e^{-i\omega t}$$

$$R(t) = R^* + \varepsilon \cdot (\tilde{R}_1 - i\hat{R}_1)e^{-i\omega t}$$

$$V(t) = V^* + \varepsilon \cdot (\tilde{V}_1 - i\hat{V}_1)e^{-i\omega t}$$

The Full SIRUV and comparison with SIR model

Combining the results for $e^{i\omega t}$ and $e^{-i\omega t}$ we obtain the real response of each variable.

$$I(t) = I^* + \varepsilon \cdot A_I \cdot cos\left(\omega\left(t + \varphi_I\right)\right)$$

$$R(t) = R^* + \varepsilon \cdot A_R \cdot cos\left(\omega\left(t + \varphi_R\right)\right)$$

$$V(t) = V^* + \varepsilon \cdot A_V \cdot cos\left(\omega\left(t + \varphi_V\right)\right)$$

with the real amplitude A and real phase φ calculated from the complex amplitude, via

$$A_I = 2\sqrt{\widetilde{I}_1^2 + \widehat{I}_1^2} \qquad \varphi_I = \frac{1}{\omega} arctan\left(\frac{\widehat{I}_1}{\widetilde{I}_1}\right)$$

$$A_R = 2\sqrt{\widetilde{R}_1^2 + \widehat{R}_1^2} \qquad \varphi_R = \frac{1}{\omega} arctan\left(\frac{\widehat{R}_1}{\widetilde{R}_1}\right)$$

$$A_V = 2\sqrt{\widetilde{V}_1^2 + \widehat{V}_1^2} \qquad \varphi_V = \frac{1}{\omega} arctan\left(\frac{\widehat{V}_1}{\widetilde{V}_1}\right)$$

The Full SIRUV and comparison with SIR model

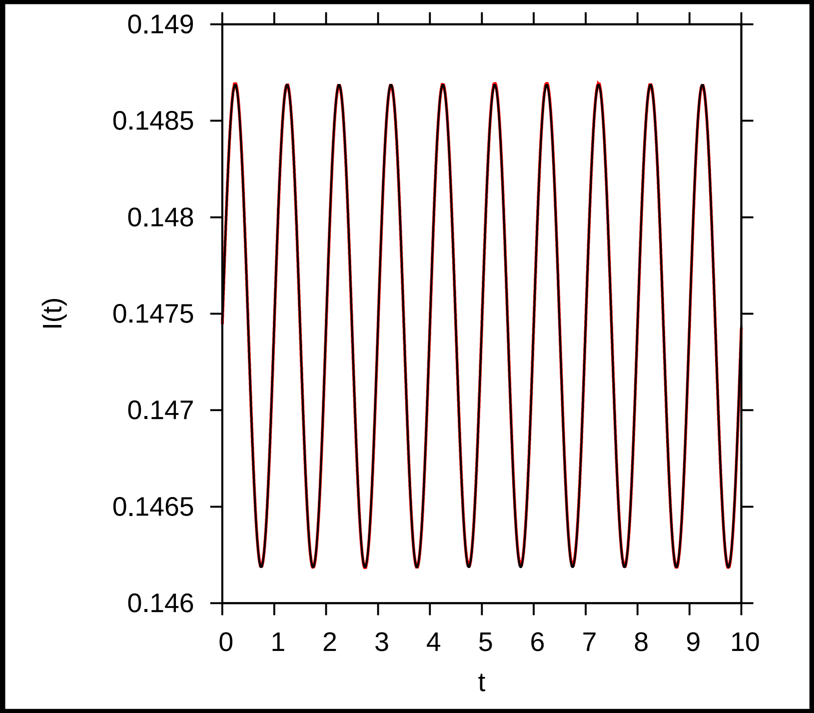
Comparing the amplitude and the phase numerically we have:

SIR	$SIRUV$
$\varepsilon A_I = 0.001955$	$\varepsilon A_I = 0.000774$
$\varphi_I = -0.248730$	$\varphi_I = -0.260094$

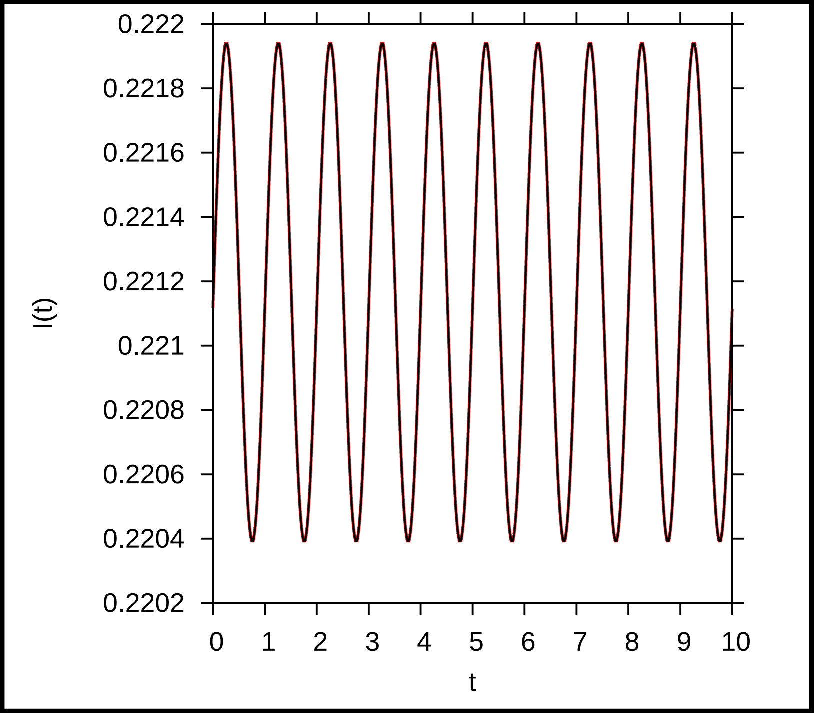
Essencially, the only difference is in the amplitude for both models.

The Full SIRUV and comparison with SIR model

SIR

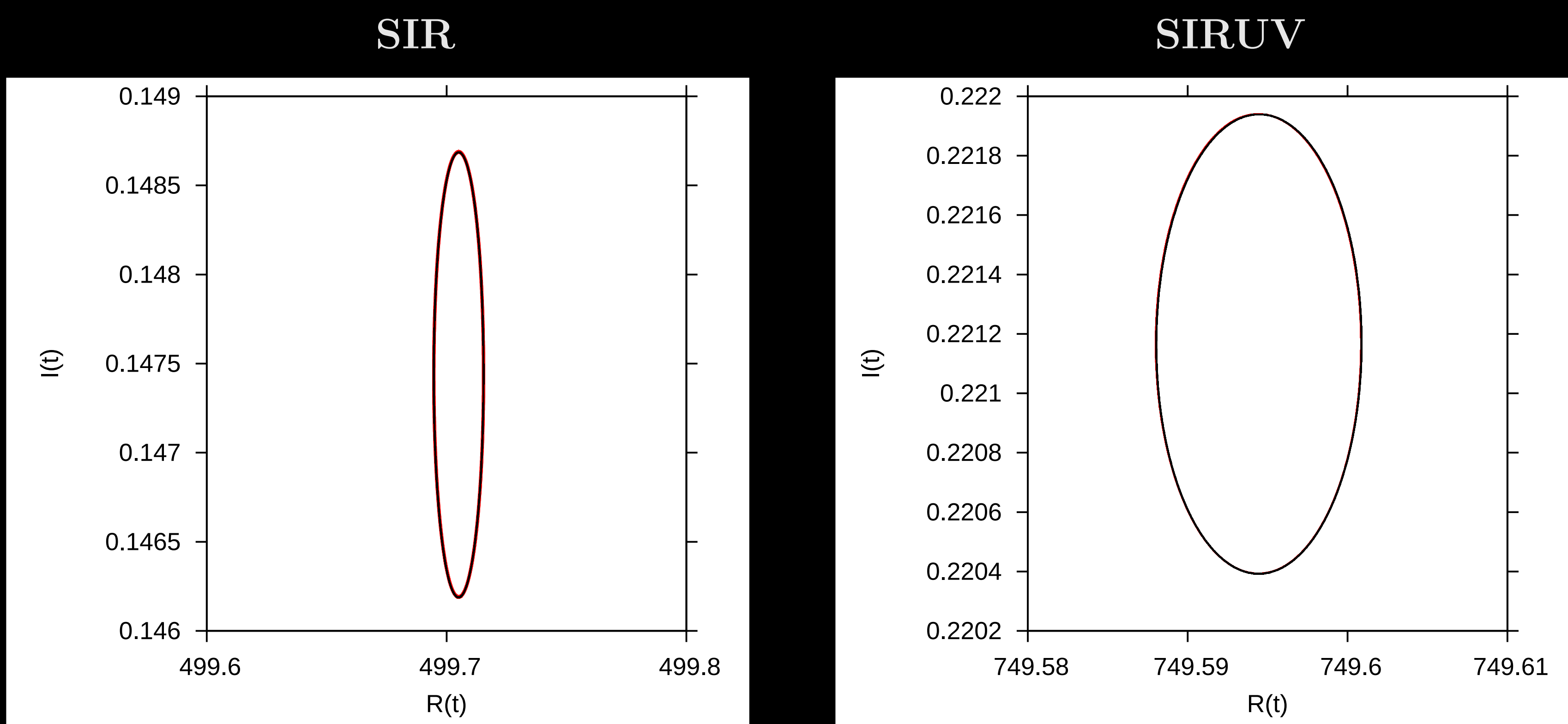


SIRUV



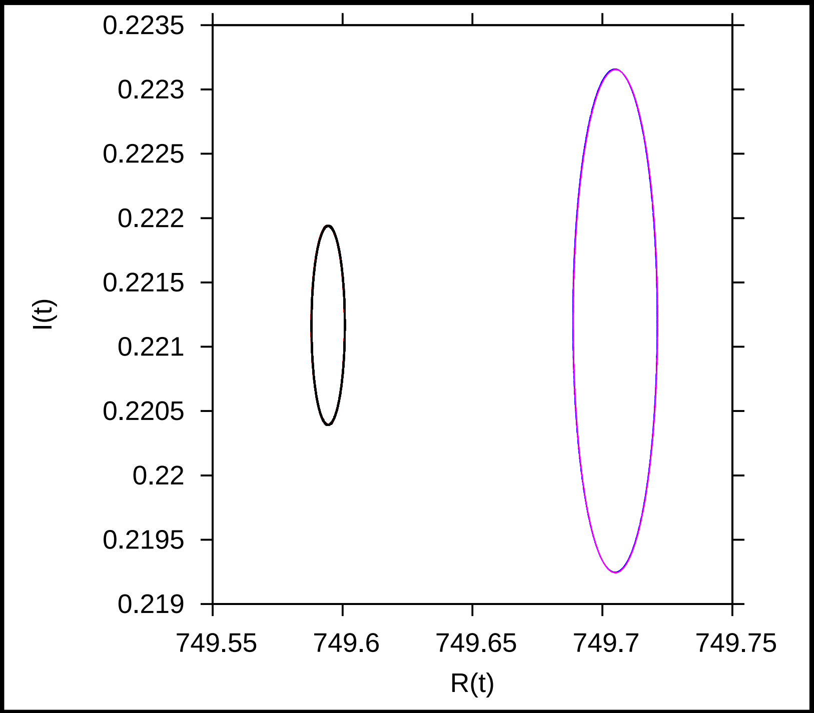
Parameters: $\gamma = \frac{365}{7} d^{-1}$, $\beta = 2\gamma$, $\mu = \frac{1}{65} y^{-1}$, $\nu = \frac{365}{10} d^{-1}$, $\vartheta = 2\nu$ and $\eta = 0.001$

The Full SIRUV and comparison with SIR model



The Full SIRUV and comparison with SIR model

We can join both models in the same plot and replace β of SIR by $\frac{\vartheta}{\nu}\beta$ in order to have the stationary states approximated.



SIRUV **SIR**

The Full SIRUV and comparison with SIR model

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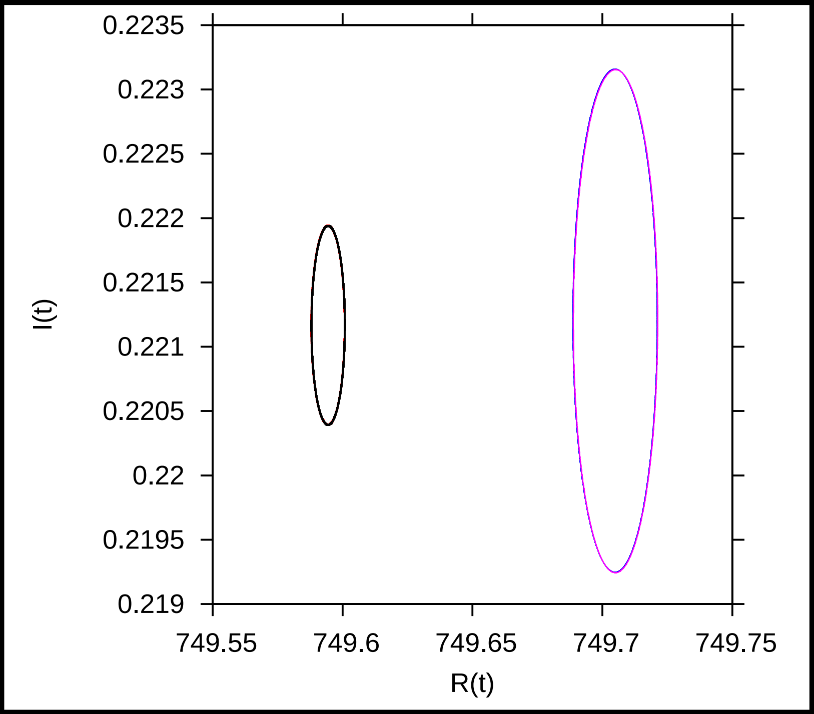
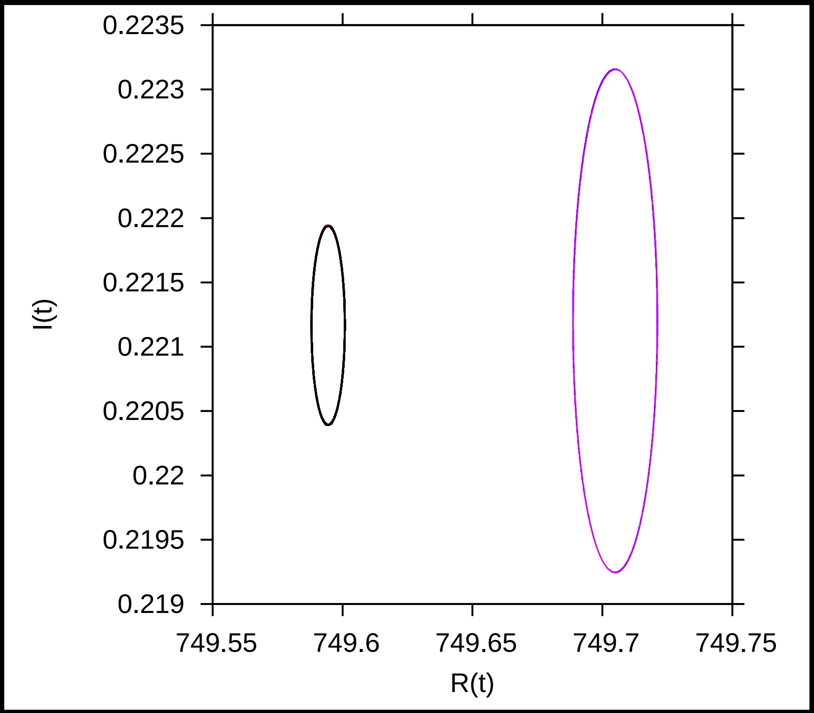


SIRUV **SIR**

Comparing the two models we can say that are not such different, the diferences in the amplitudes of I are in order of 0.001, so it is basically the same for both models.

The Full SIRUV and comparison with SIR model

We can join both models in the same plot and replace β of SIR by $\frac{\vartheta}{\nu}\beta$ in order to have the stationary states approximated.



SIRUV SIR

Comparing the two models we can say that are not such different, the diferences in the amplitudes of I are in order of 0.001, so it is basically the same for both models.

So, once more, we can say that mosquitos do not add any information to models.

Thank you for your attention!

"I try to be considerate and always sterilize my stinger before biting into a new victim."