Study of a stochastic epidemic model with the BSDE approach

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Abstract

The interest is focused on studying the number of cases of infection in a population of K computers using the block-structured state-dependent event (BSDE) approach. The BSDE approach provides the possibility of considering non-exponential models with correlated flows, but keeping tractable the dimensionality of the underlying Markov chain.

The work is inspired in the study of the number of recovered and infected individuals in the SIS scalar stochastic epidemic models. It considers an extension of a flow of external infections as well as the possibility of dealing with batch infection taking place when the infection initiated at an individual computer is immediately propagated to other computers of the network.

The model

Scalar version

SIS MODEL: $K \equiv$ size of the population, $S(t) \equiv$ number of susceptible computers at time t, $I(t) \equiv$ number of infected computers at time t. $S(t) + I(t) = K$.

The evolution of the epidemic is described by the process $\{I(t);\ t \geq 0\}$, with state space $\{0,\ldots,K\}$. It is modeled as a birth and death process with rates $\lambda_i = \frac{\beta}{K} i(K-i),\ 0 \leq i \leq K, \quad \mu_i = \gamma i,\ 0 \leq i \leq K,$

β is the contact rate from an infected computer to a susceptible computer and γ is the individual recovery rate.

Two extensions:

1. The existence of a permanent external source of infection, with rate δ. $\lambda_i = \left(\delta + \frac{\beta i}{K}\right)(K-i),\ 0 \leq i \leq K.$
2. Batch infections: Given that $I(t)=i$, the next infection takes places after an exponentially distributed time with rate λ_i causing the infection of j susceptible computers with probability g_{ij}, for $0 \leq i \leq K$-1 and $1 \leq j \leq K$-i.

BSDE model

CTMC: $(X, \vec{Y}) = \{(X_t, Y_t^1, Y_t^2);\ t \geq 0\}$

Number of infected computers at time t. — Infection phase at time t. — Recovery phase at time t.

The infinitesimal generator $\mathbf{Q}$ of this CTMC has a finite level-dependent upper Hessenberg structure with matrix blocks:

$$\mathbf{Q}_{i,i-1} = \mathbf{I}_M \otimes \overline{\mathbf{R}}_i^1,\ 1 \leq i \leq K,$$
$$\mathbf{Q}_{ij} = \overline{\mathbf{I}}_i^{j-i} \otimes \mathbf{I}_N,\ 0 \leq i \leq K-1,\ i+1 \leq j \leq K,$$
$$\mathbf{Q}_{ii} = (1-\delta_{iK})(\overline{\mathbf{I}}_i^0 \otimes \mathbf{I}_N) + (1-\delta_{i0})(\mathbf{I}_M \otimes \overline{\mathbf{R}}_i^0),\ 0 \leq i \leq K$$

$$\overline{I}_i^0(m;m) = -\lambda_{im}^I,\quad \overline{I}_i^0(m;m') = \lambda_{im}^I I_i^0(m;m'),\ m' \neq m,\quad \overline{I}_i^j(m;m') = \lambda_{im}^I I_i^j(m;m'),\ 1 \leq j \leq K-i,$$

$$\overline{R}_i^0(n;n) = -\lambda_{in}^R,\quad \overline{R}_i^0(n;n') = \lambda_{in}^R R_i^0(n;n'),\ n' \neq n,\quad \overline{R}_i^1(n;n') = \lambda_{in}^R R_i^1(n;n').$$

λ_{im}^I : Rate of an exponential sojourn time which ends when an infection takes place (with or without phase change) or simply when the infection phase changes.

$I_i^j(m;m')$: Probability that the effective contact causes the infection of j susceptible computers and a transition from the phase m to the phase m', given that $X_t = i$.

Particularizations of $\overline{\mathbf{I}}_i^j$ and $\overline{\mathbf{R}}_i^j$:

$(\mathbf{D}_0^I, \mathbf{D}_1^I)$, $(\mathbf{D}_0^R, \mathbf{D}_1^R)$: Characteristic matrices of two auxiliary MAPs of orders M and N, with fundamental rates λ^I and λ^R.

$$\overline{\mathbf{I}}_i^0 = \frac{\lambda_i}{\lambda^I}\mathbf{D}_0^I,\ 0 \leq i \leq K-1,\quad \overline{\mathbf{I}}_i^{j-i} = \frac{\lambda_i g_{i,j-i}}{\lambda^I}\mathbf{D}_1^I,\ 0 \leq i \leq K-1,\ i+1 \leq j \leq K,\quad \overline{\mathbf{R}}_i^0 = \frac{\mu_i}{\lambda^R}\mathbf{D}_0^R,\ 1 \leq i \leq K,\quad \overline{\mathbf{R}}_i^1 = \frac{\mu_i}{\lambda^R}\mathbf{D}_1^R,\ 1 \leq i \leq K.$$

Number of infections until the extinction time

Probability mass function

$N_{imn} \equiv$ Number of infections until the first time at which $X_t = 0$, given that the current system state is (i, m, n). (Every computer can be infected several times before the extinction time)

$x_{imn}^k = P\{N_{imn} = k\}$, for $(i,m,n) \in S_{X,\vec{Y}}$ and $k \geq 0$. These probabilities satisfy:

$$x_{0mn}^k = \delta_{k0},\ 1 \leq m \leq M,\ 1 \leq n \leq N,\ k \geq 0,$$

$$x_{imn}^k = \frac{\lambda_{im}^I}{\lambda_{imn}}\left(\sum_{\substack{m'=1\\ m' \neq m}}^{M} I_i^0(m;m')x_{im'n}^k + (1-\delta_{k0})(1-\delta_{iK})\sum_{j=1}^{\min\{k,K-i\}}\sum_{m'=1}^{M} I_i^j(m;m')x_{i+j,m'n}^{k-j}\right) + \frac{\lambda_{in}^R}{\lambda_{imn}}\left(\sum_{\substack{n'=1\\ n' \neq n}}^{N} R_i^0(n;n')x_{imn'}^k + \sum_{n'=1}^{N} R_i^1(n;n')x_{i-1,mn'}^k\right),\quad 1 \leq i \leq K,\ 1 \leq m \leq M,\ 1 \leq n \leq N,\ k \geq 0.$$

Generating function

$\Phi_{imn}(z) = E[z^{N_{imn}}],\ (i,m,n) \in S_{X,\vec{Y}},\ |z| \leq 1.$ $\quad \Phi_{0mn}(z) = 1,\ 1 \leq m \leq M,\ 1 \leq n \leq N,$

$$\Phi_{imn}(z) = \frac{\lambda_{im}^I}{\lambda_{imn}}\left(\sum_{\substack{m'=1\\ m' \neq m}}^{M} I_i^0(m;m')\Phi_{im'n}(z) + (1-\delta_{iK})\sum_{j=1}^{K-i} z^j \sum_{m'=1}^{M} I_i^j(m;m')\Phi_{i+j,m'n}(z)\right) + \frac{\lambda_{in}^R}{\lambda_{imn}}\left(\sum_{\substack{n'=1\\ n' \neq n}}^{N} R_i^0(n;n')\Phi_{imn'}(z) + \sum_{n'=1}^{N} R_i^1(n;n')\Phi_{i-1,mn'}(z)\right),$$

$1 \leq i \leq K,\ 1 \leq m \leq M,\ 1 \leq n \leq N.$ This system can be written in matrix form as $\mathbf{Q}(z)\mathbf{\Phi}(z) = \mathbf{b}.$

Moments

$$M_{imn}^0 = P\{N_{imn} < \infty\},\ M_{imn}^k = E[N_{imn}(N_{imn}-1)\cdots(N_{imn}-k+1)],\ k \geq 1,\ (i,m,n) \in S_{X,\mathbf{Y}}.$$

$$\mathbf{Q}(1)\mathbf{M}^k = -\sum_{r=1}^{\min\{k,K-1\}} \binom{k}{r}\mathbf{Q}^{(r)}(1)\mathbf{M}^{k-r},\ k \geq 1,$$ where $\mathbf{M}^k = (\mathbf{M}_0^k,\ldots,\mathbf{M}_K^k)'$, $\mathbf{M}_i^k = (M_{i11}^k,\ldots,M_{iMN}^k)'$, for $0 \leq i \leq K$.

$$\mathbf{Q}^{(r)}(1) = \frac{d^r}{dz^r}\mathbf{Q}(z)\Big|_{z=1}.$$

Number of infections during (0,t]

$CTMC\ (X, \vec{Y}, N) = \{(X_t, Y_t^1, Y_t^2, N_t);\ t \geq 0\}$. N_t is the number of infections in the interval (0, t], given that the initial state is $(i_0, m_0, n_0, 0)$.

Transient probabilities are defined by $p_{imnk}(t) = P\{X_t = i,\ Y_t^1 = m,\ Y_t^2 = n,\ N_t = k\}$,

and the initial probability $p_{imnk}(0) = \delta_{(i,m,n,k)(i_0,m_0,n_0,0)}$.

Laplace transform of $p_{imnk}(t)$: $\widetilde{p}_{imnk}(s) = \int_0^\infty e^{-st}p_{imnk}(t)dt,\quad \mathrm{Re}(s) \geq 0.$ $\quad \widetilde{\mathbf{p}}_k(s)\widetilde{\mathbf{Q}}(s) = \widetilde{\mathbf{b}}_k,\ k \geq 0,$

$$\widetilde{\mathbf{Q}}_{ij}(s) = \begin{cases} \mathbf{Q}_{ij} - s\mathbf{I}_g, & \text{if } 0 \leq i \leq K,\ i = j,\\ \mathbf{Q}_{ij}, & \text{if } 1 \leq i \leq K,\ j = i-1,\\ \mathbf{0}_{g \times g}, & \text{otherwise,}\end{cases}\qquad \widetilde{\mathbf{b}}_{0k} = \begin{cases} -\delta_{0i_0}\mathbf{e}_g'((m_0-1)N+n_0), & \text{if } k = 0,\\ \mathbf{0}_g', & \text{if } k \geq 1,\end{cases}$$

and for $1 \leq i \leq K$

$$\widetilde{\mathbf{b}}_{ik} = \begin{cases} -\delta_{ii_0}\mathbf{e}_g'((m_0-1)N+n_0), & \text{if } k = 0,\\ -\sum_{j=1}^{\min\{i,k\}} \widetilde{\mathbf{p}}_{i-j,k-j}(s)\mathbf{Q}_{i-j,i}, & \text{if } 1 \leq k \leq K,\\ -\sum_{j=1}^{i} \widetilde{\mathbf{p}}_{i-j,k-j}(s)\mathbf{Q}_{i-j,i}, & \text{if } k \geq K+1.\end{cases}$$

Marginal distribution $p_{\ldots k}(t) = P\{N_t = k\}$, for $k \geq 0$. Inverting numerically the sum $\sum_{(i,m,n) \in S_{X,\vec{Y}}} \widetilde{p}_{imnk}(s)$.

Moments of N_t $\quad m_r(t) = \sum_{k=0}^{\infty} k^r p_{\ldots k}(t) = \sum_{(i,m,n) \in S_{X,\vec{Y}}} m_{imn}^r(t),\ r \geq 1,$ where $m_{imn}^r(t) = \sum_{k=0}^{\infty} k^r p_{imnk}(t).$

Laplace transform: $\widetilde{m}_{imn}^r(s) = \int_0^\infty e^{-st}m_{imn}^r(t)dt$

$$\widetilde{\mathbf{m}}^r(s)(\mathbf{Q} - s\mathbf{I}_{(K+1)g}) = \widetilde{\mathbf{c}}^r,\ r \in \{0,1,2\},\qquad \widetilde{\mathbf{c}}_i^r = \begin{cases} -\delta_{ii_0}\mathbf{e}_g'((m_0-1)N+n_0), & \text{if } r = 0,\\ -(1-\delta_{i0})\sum_{j=1}^{i} j\widetilde{\mathbf{m}}_{i-j}^0(s)\mathbf{Q}_{i-j,i}, & \text{if } r = 1,\\ -(1-\delta_{i0})\sum_{j=1}^{i}\left(j^2\widetilde{\mathbf{m}}_{i-j}^0(s) + 2j\widetilde{\mathbf{m}}_{i-j}^1(s)\right)\mathbf{Q}_{i-j,i}, & \text{if } r = 2.\end{cases}$$

Numerical examples

(*i*) Exponential kernel: $\mathbf{D}_0^{Ex} = -1,\ \mathbf{D}_1^{Ex} = 1.$

(*ii*) Erlang kernel: $\mathbf{D}_0^{Er} = \begin{pmatrix} -3 & 3 & 0\\ 0 & -3 & 3\\ 0 & 0 & -3\end{pmatrix},\ \mathbf{D}_1^{Er} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 3 & 0 & 0\end{pmatrix}.$

(*iii*) Hiperexponential kernel: $\mathbf{D}_0^{H} = \begin{pmatrix} -1.9 & 0\\ 0 & -0.19\end{pmatrix},\ \mathbf{D}_1^{H} = \begin{pmatrix} 1.71 & 0.19\\ 0.171 & 0.019\end{pmatrix}.$

(*iv*) MAP kernel with negative correlation (-0.48890): $\mathbf{D}_0^- = \begin{pmatrix} -1.00221 & 1.00221 & 0\\ 0 & -1.00221 & 0\\ 0 & 0 & -225.75\end{pmatrix},\ \mathbf{D}_1^- = \begin{pmatrix} 0 & 0 & 0\\ 0.01002 & 0 & 0.99219\\ 223.4925 & 0 & 2.2575\end{pmatrix}.$

(*v*) MAP kernel with positive correlation (0.43482): $\mathbf{D}_0^+ = \begin{pmatrix} -0.87478 & 0.87478 & 0\\ 0 & -0.87478 & 0\\ 0 & 0 & -94.76811\end{pmatrix},\ \mathbf{D}_1^+ = \begin{pmatrix} 0 & 0 & 0\\ 0.78730 & 0 & 0.08748\\ 7.28985 & 0 & 87.47826\end{pmatrix}.$

(γ,δ,θ) = (1.0,0.01,0.25)

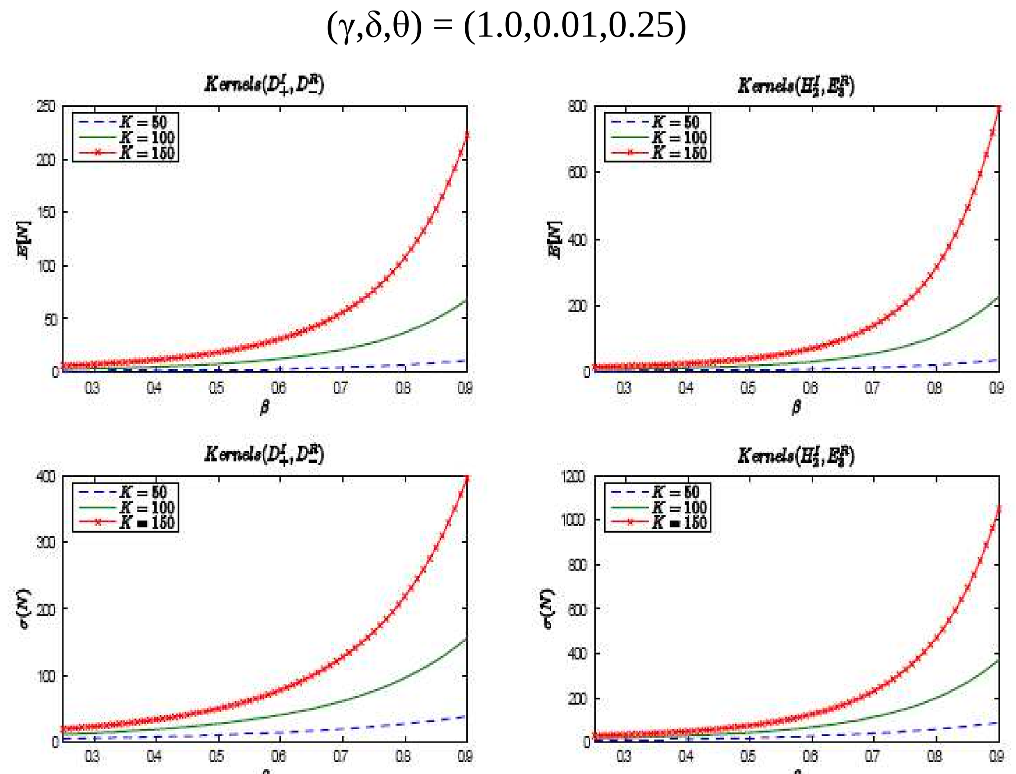

Figure 1. $E[N]$ and σ(N) as a function of β and K

(γ,δ,β) = (1.0,0.01,0.25)
K = 100

		(Exp^I, Exp^R)	(H_2^I, E_3^R)	(D_-^I, D^R)	(D_+^I, E_3^R)
θ = 0.2	$E[N]$	19.08	16.25	6.44	6.86
	σ(N)	30.65	35.61	23.40	24.64
θ = 0.3	$E[N]$	36.51	24.91	10.02	10.94
	σ(N)	55.51	53.39	33.41	35.70
θ = 0.4	$E[N]$	87.69	42.98	17.19	19.32
	σ(N)	122.93	88.21	51.28	55.84
θ = 0.5	$E[N]$	307.89	87.16	33.98	39.58
	σ(N)	382.22	165.38	87.42	97.64
θ = 0.6	$E[N]$	2030.63	216.50	83.83	102.48
	σ(N)	2189.27	362.01	175.94	204.26
θ = 0.7	$E[N]$	36482.59	666.24	302.01	399.46
	σ(N)	36745.85	941.93	478.98	598.59
θ = 0.8	$E[N]$	2575690.2	2412.78	2285.95	3588.12
	σ(N)	2576022.7	2889.69	2625.56	3971.56

Table 1. $E[N]$ and σ(N) as a function of θ and the kernel choice

(K, β, γ, δ, θ) = (50,0.75,1.0,0.01,0.25)

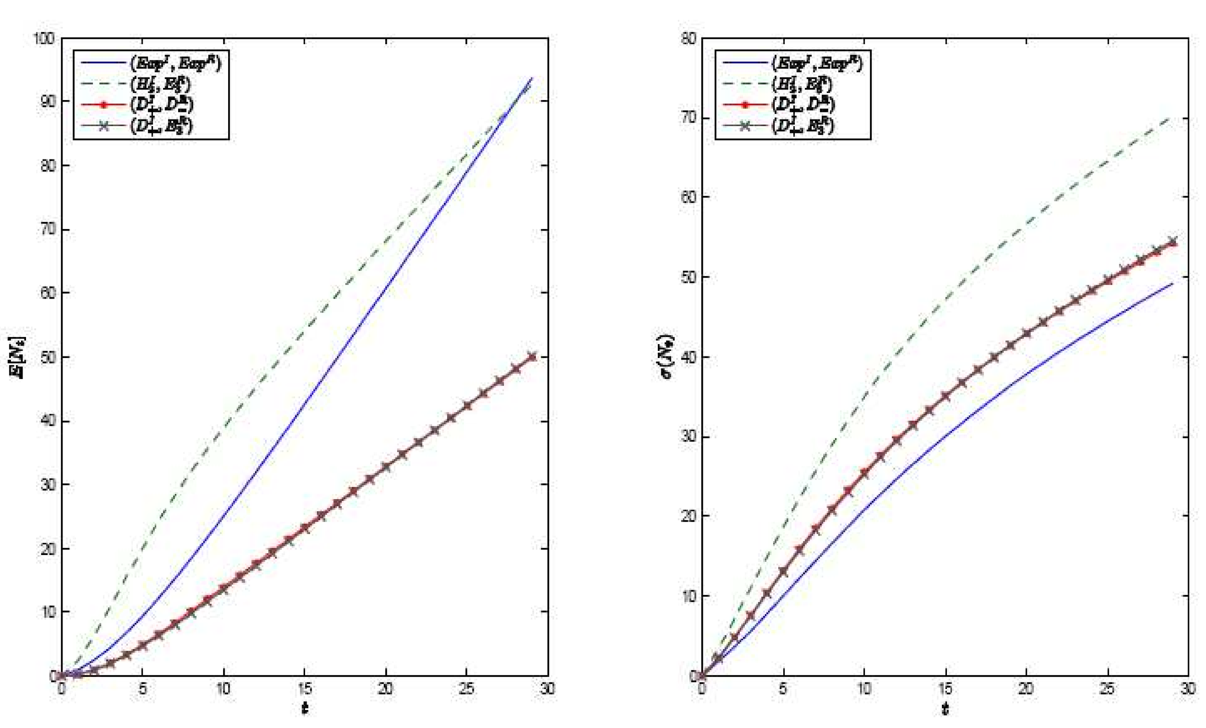

Figure 2. $E[N_t]$ and σ(N_t) as a function of the kernel choice