On a Exponential Decay of the Solution for a Stochastic Coupled System of Reaction-Diff usion of Nonlocal Type

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Outline of the talk

1. Introduction
2. Notation formulation of the problem
3. Existence and uniqueness of solution
4. Asymptotic behavior

Introduction

In this talk I study the following initial-boundary value problem involving a stochastic nonlinear parabolic equation of nonlocal type

$$\begin{cases} u_t - a(\int_D u \; dx)\Delta u = g_1(v) + f_1(u,v)\dfrac{\partial W_1}{\partial t} \text{ on } D \times]0,\infty[\,, \\ v_t - a(\int_D v \; dx)\Delta v = g_2(u) + f_2(u,v)\dfrac{\partial W_2}{\partial t} \text{ on } D \times]0,\infty[\,, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \text{ in } D, \\ (u,v) \quad = (0,0) \text{ on } \partial D \times]0,\infty[\qquad . \end{cases} \tag{1}$$

where D is a bounded open subset of $\mathbb{R}^n$ with boundary ∂D, $n \geq 1$, $a = a(s)$ is a continuous function with Lipschitz's constant L such that $0 < p \leq a(s) \leq P$ where p and P are constants, $(W_1, W_2)_{t \in [0,\infty[}$ is a two dimensional Wiener process, the maps $f_i : L^2(D) \times L^2(D) \to L^2(D)$, $g_i : L^2(D) \to L^2(D)$, with $i = 1, 2$ satisfies the following conditions

i. $\|f_i(u_1, v_1) - f_i(u_2, v_2)\|^2 \leq J\left(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2\right)$,
ii. $f_i(0, 0) = 0$ for $t \in [0, \infty[$,
iii. $\|g_i(v_1) - g_i(v_2)\|^2 \leq K\left(\|v_1 - v_2\|^2\right)$,
iv. $g_i(0) = 0$ for $t \in [0, T]$,

with $J, K > 0$.

I consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\left(\mathcal{F}_{t\in[0,T]}\right)$ is a right continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}-$null sets, $E(X)$ denote the mathematical expectation of the random variable X, we abreviate a.s. for *almost surely* $\omega \in \Omega$. We write $\mathcal{L}$ for the Lebesgue measure on $\mathbb{T} := [0, \infty[$.
Let $H^s(D)$ denote the usual Sobolev space of order s with norm $\|\cdot\|_s$, and inner product $(\cdot,\cdot)_s$, $H_0^1(D)$ the Sobolev space of order 1 with zero boundary condition with dual space $H^{-1}(D)$, $H^0(D) = L^2(D)$ with norm $\|\cdot\| := \|\cdot\|_0$ and inner product $(\cdot,\cdot) := (\cdot,\cdot)_0$.

Let B a Banach space with norm $\| \cdot \|_B$, $\mathcal{B}(B)$ denote the Borel $\sigma-$algebra of B. The space $L^2(\Omega, \mathbb{T}; B)$ is the set of all $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})-$measurable process $u : \Omega \times \mathbb{T} \to B$ which are $\mathcal{F}_t-$ adapted and $E(\int_0^T \|u\|_B^2 dt) < \infty$. Analogously the space $L^\infty(\Omega, \mathbb{T}; \mathbb{R})$ is the set of all $\mathcal{F} \otimes \mathcal{B}(\mathbb{T})-$measurable process $u : \quad \times \mathbb{T} \to \mathbb{R}$ which are $\mathcal{F}_t-$ adapted and for almost everywhere $(\omega, t) \in \quad \times \mathbb{T}$ bounded. In this work $(W)_{t \in \mathbb{T}}$ is real Wiener process $\mathcal{F}_t-$adapted. Various constants will be denoted by c and $c(D) := \int_D dx$.
We define the map $\mathcal{A} : H_0^1(D) \to H^{-1}(D)$ by

$$\langle \mathcal{A}u, \eta \rangle = a \left(\int_D u \, dx \right) (\nabla u, \nabla \eta)$$

for $\eta \in H_0^1(D)$.

I assume that

$$(\nabla u, \nabla u) = \|u\|_1^2 \quad \text{for all } u \in H_0^1(D).$$

Let $u_0,\ v_0$ be random variables L^2-valued, $\mathcal{F}_0-$measurable such that $E(\|u_0\|^2 + \|v_0\|^2) < \infty$.
In this work i mean that the stochastic process (u, v) is a solution of the problem (1) in the following sense:

Definition

The stochastic process $(u(t), v(t))_{t\in\mathbb{T}} \in L^2(\Omega, \mathbb{T}; {H_0}^1(D)) \times L^2(\Omega, \mathbb{T}; {H_0}^1(D))$ with a.s. sample paths continuous in $L^2(D) \times L^2(D)$, is a solution of (1) if it satisfies the equation:

$$
\begin{aligned}
(u(t),\eta) + \int_0^t \langle \mathcal{A}u,\eta\rangle(s)ds &= (u_0,\eta) + \int_0^t (f_1(u(s),v(s)),\eta)dW_1(s) \\
&+ \int_0^t (g_1(v),\eta)ds \\
(v(t),\xi) + \int_0^t \langle \mathcal{A}v,\xi\rangle(s)ds &= (v_0,\xi) + \int_0^t (f_2(u(s),v(s)),\xi)dW_2(s) \\
&+ \int_0^t (g_2(u),\xi)ds
\end{aligned} \tag{2}
$$

a.s. for all $\eta,\ \xi \in H_0^1(D)$ and $t \in \mathbb{T}$, where the stochastic integral is in the Itô sense.

I mean uniqueness in the sense of indistinguishable.

Existence and Uniqueness of Strong Solution

Theorem (Existence and Uniqueness of Strong Solution)

Suppose $p > J + \frac{K+1}{2}$ *and* $p > 3K$. *The problem* (1)*has a solution, which is unique and has a.s. sample paths continuous in* $L^2(D) \times L^2(D)$.

Asymptotic behavior

Theorem

Suppose that $2p - 2J - K - \frac{1}{2} > 0$. *Then the solution* $(u(t), v(t))_{t \in \mathbb{T}}$ *obtained in Theorem of Existence and Uniqueness of Strong Solution satisfies*

$$\lim_{t \to \infty} E\|u(t)\| \leq (\|u_0\| + \|v_0\|). \tag{3}$$

References

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Thank You!