Universidade Nova de Lisboa
OMNIS CIVITAS CONTRA SE DIVISA NON STABIT
Faculdade de Ciências e Tecnologia

UNINOVA
Instituto de Desenvolvimento de Novas Tecnologias

# FRACTIONAL?
# WHERE?

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# FRACTIONAL? CALCULUS?

## What is it?

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## What is it?

## Non integer order derivatives.

# FRACTIONAL? CALCULUS?

What is it?

**Non integer order derivatives.**

**Power law responses.**

# FRACTIONAL? CALCULUS?

What is it?

**Non integer order derivatives.**

**Power law responses.**

**Fractals.**

# FRACTIONAL? CALCULUS?

## Where can we find it?

# FRACTIONAL? CALCULUS?

## Where can we find it?

## EVERYWHERE

# Contents

- *Motivation and practical examples*
- *The causal fractional derivatives*
  - *Positive integer order*
  - *Negative integer order*
  - *Real order - Grünwald-Letnikov*
  - *Properties*
- *The derivative operator as linear system*
  - *The transfer function/frequency response*
  - *The impulse response*

# Physical existence of  fractional order systems

- Wheather/climate
- Economy/finance
- Biology/Genetics
- Music
- Biomedics
- Physics
- …

# *Fractionality in Nature and Science*

- ***1/f noises***
- ***Long range processes (Economy, Hydrology)***
- ***The fractional Brownian motion***
- ***The constant phase elements***
- ***Music spectrum***
- ***Network traffic***
- ***Biological processes - Deterministic Genetic Oscillation***
- ***Heat Conduction in a Porous Medium***
- ***Geometry***

# Rule of thumb

- Self-similar
- Scale-free/Scale-invariant
- Power law
- Long range dependence (LRD)
- $1/f^{a}$ noise
- Porous media
- Particulate
- Granular
- Lossy
- Anomaly
- Disorder
- Soil, tissue, electrodes, bio, nano, network, transport, diffusion, soft matters …

# Engineering applications

***Control***

***Filtering***

***Image processing***

***System modelling – NMR, Diffusion, respiratory system, muscles, neurons***

***Calculus of variations - Optimization***

***Chaos***

***Fractals***

# *Birth and evolution*

- In the very beginning of calculus Leibnitz introduced the notation

$$\frac{d^n}{dt^n}$$

Soon he received an enquiry from L'Hôpital:

*What if n is 1/2?*

- Leibnitz's replay:

*It will lead to a paradox; a paradox from which one day useful consequences will be drawn*

# *Birth and evolution*

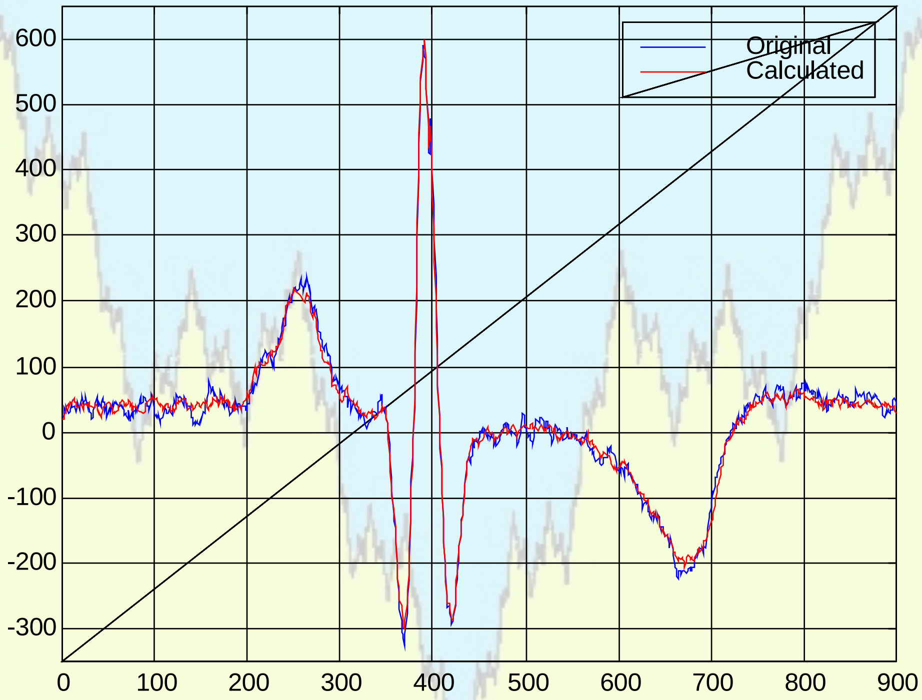
•For three centuries fractional calculus developed mainly as a pure theoretical mathematical discipline.

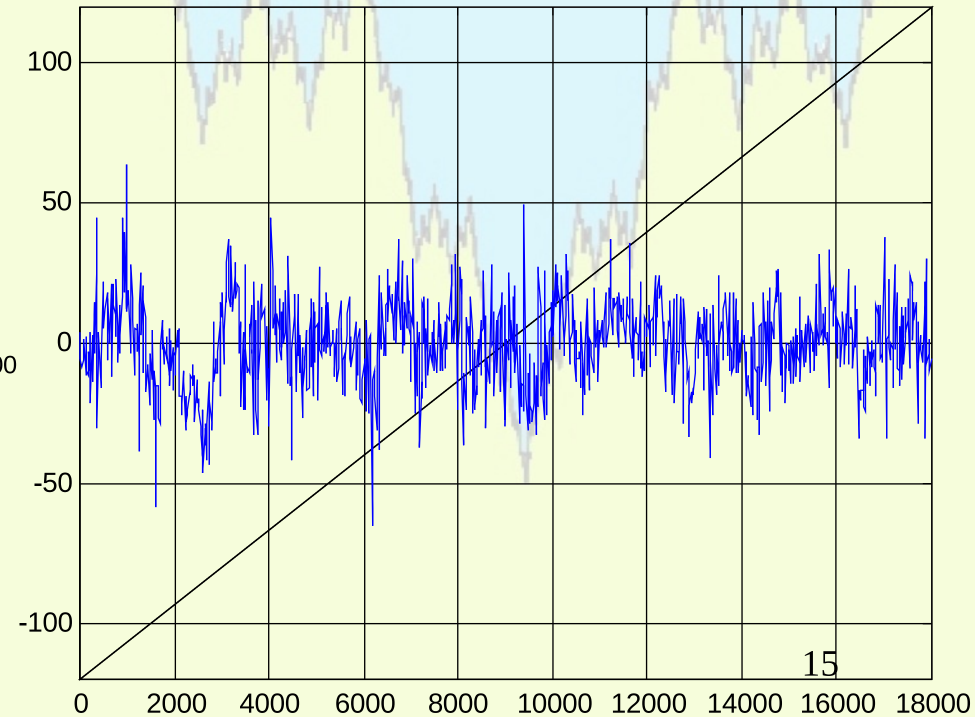
•In the last decades: description of dynamic behavior of various physical systems and real materials.

•Main reason: fractional derivatives and integrals, by sharing and unified definition as convolution integrals, provide an excellent instrument for the description of memory and hereditary properties.

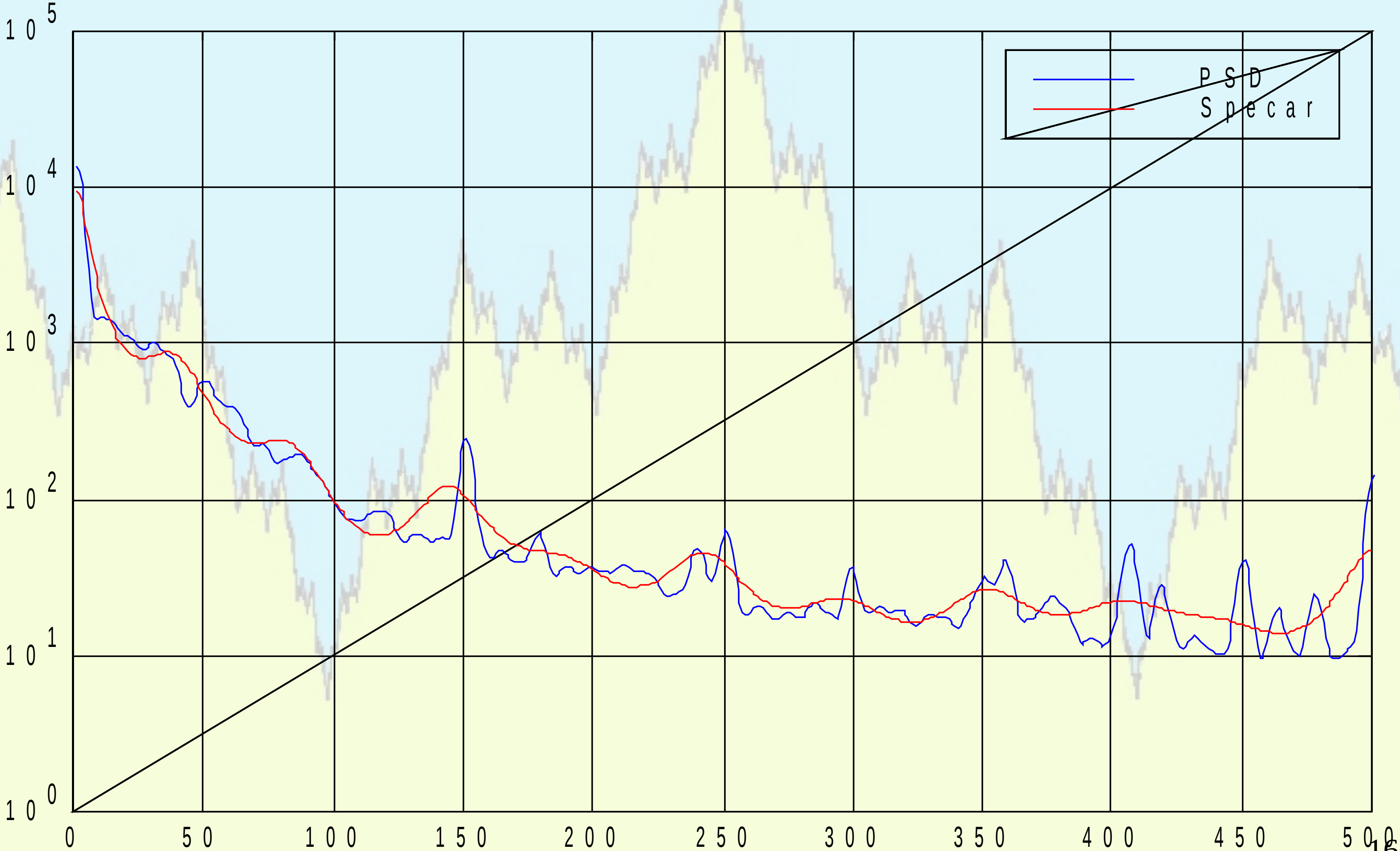
•Nowadays: electrochemistry, diffusion, probability, viscoelasticity and hereditary mechanics, control theory, and others.

# Example: ECG beat and noise



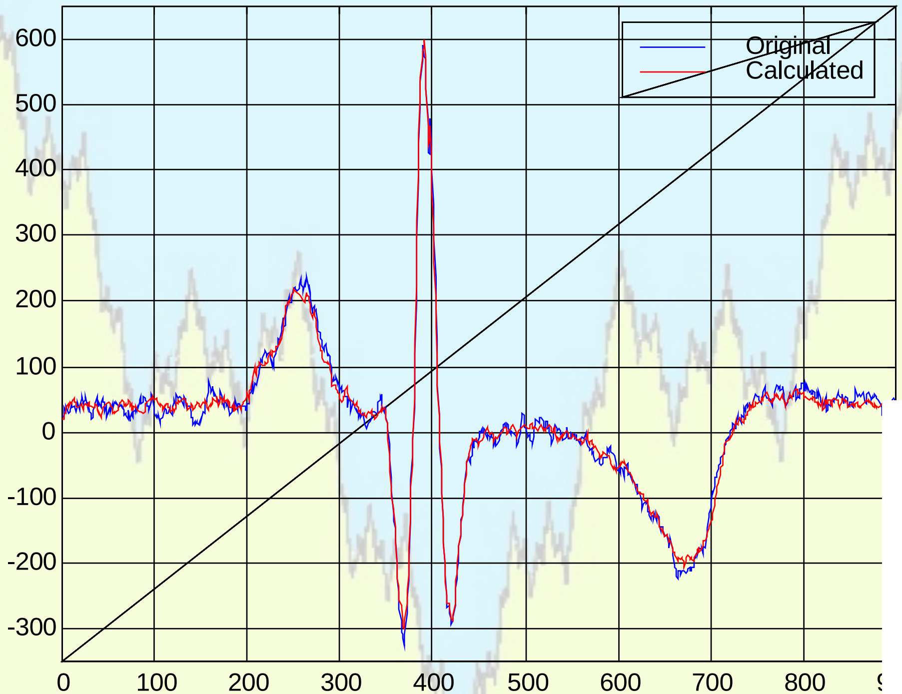



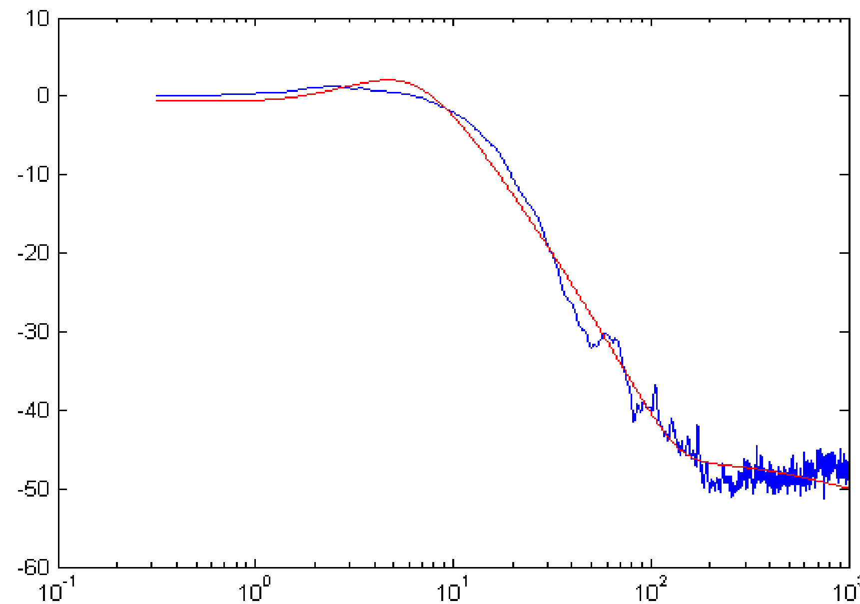
# The noise spectrum in ECG



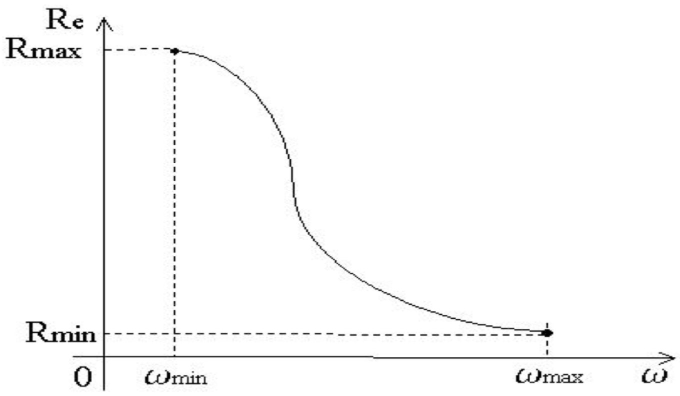

Power spectral density of ECG noise

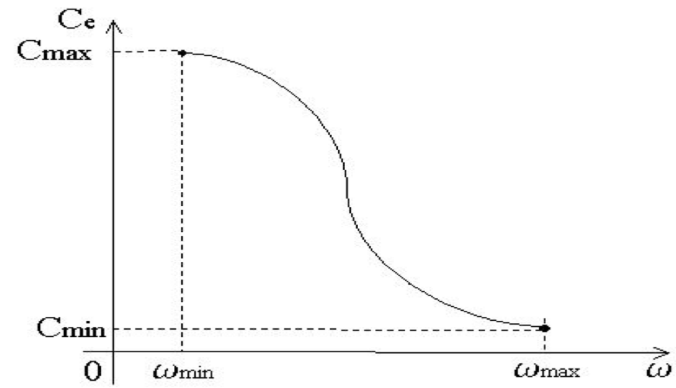
# Example: ECG beat spectrum

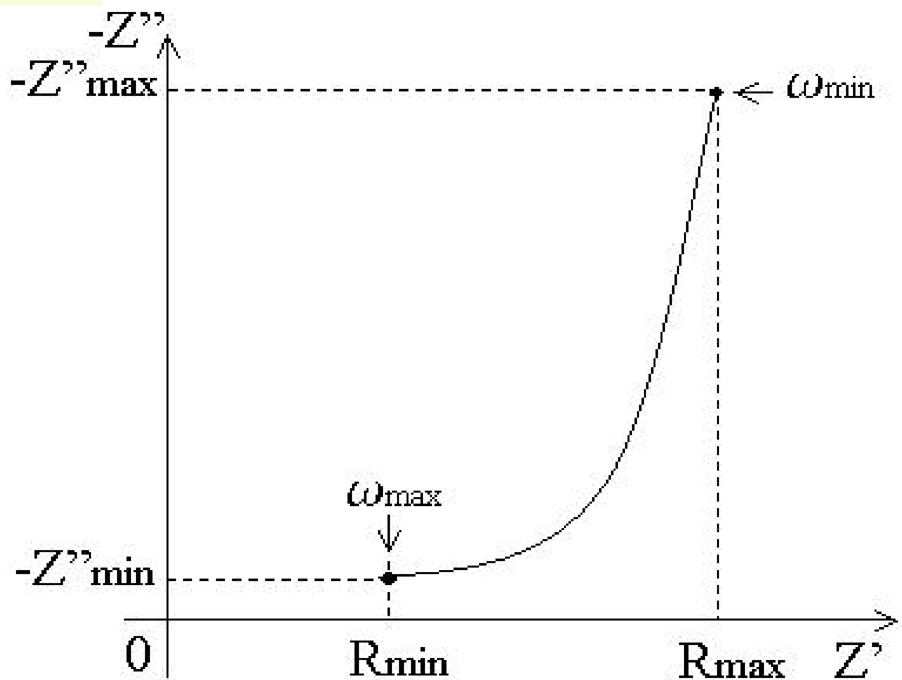


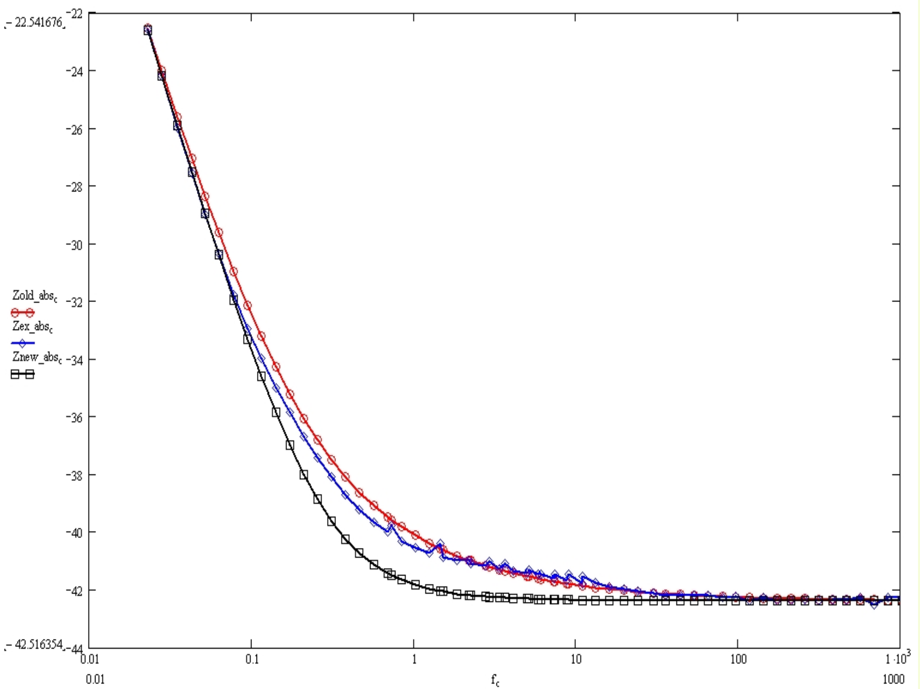



# Example: supercapacitor



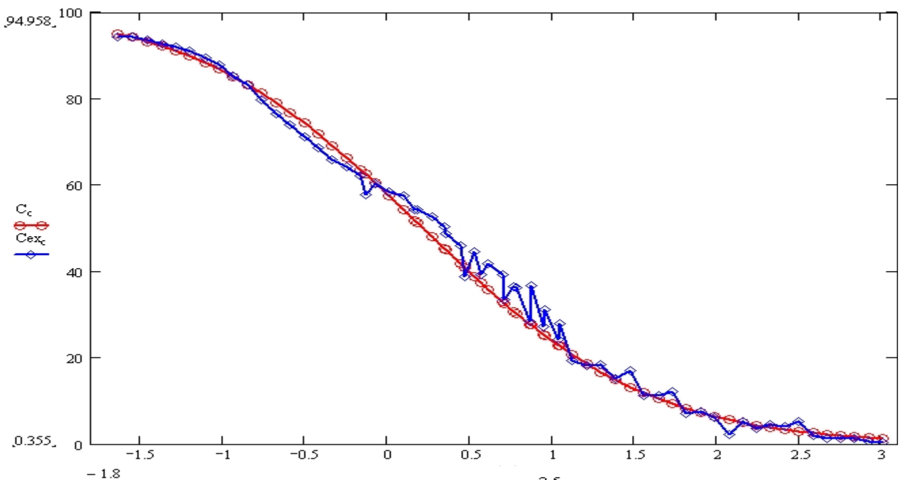



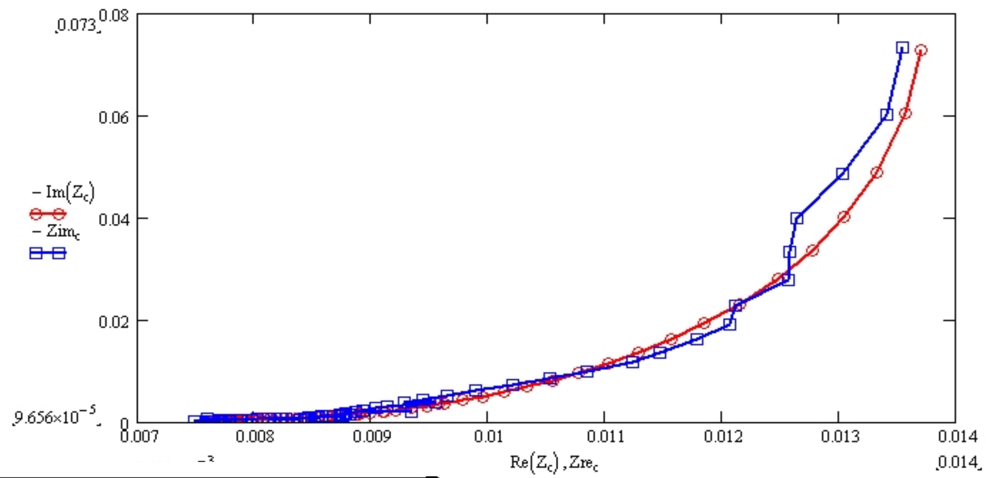



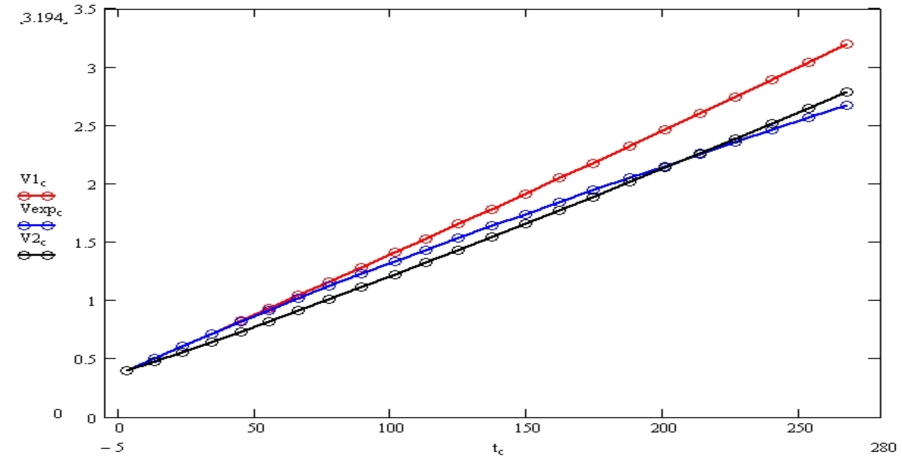



# Example: supercapacitor

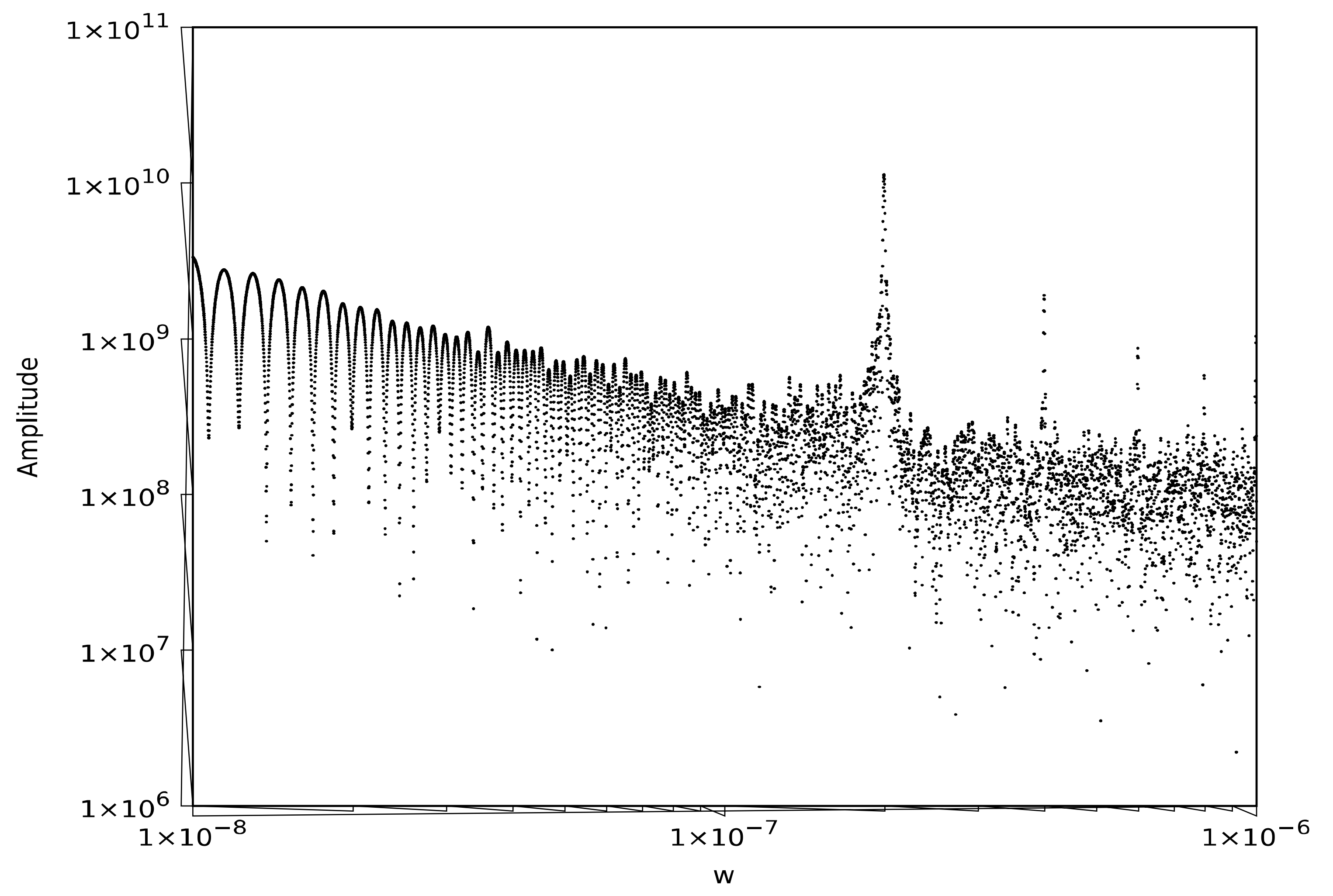
$$H(s) = 7.39\times10^{-3} + \frac{3.24\times10^{-3}}{s^{b}} + \frac{7.68\times10^{-3}}{s^{a}} + \frac{3.37\times10^{-3}}{s^{a+b}}$$

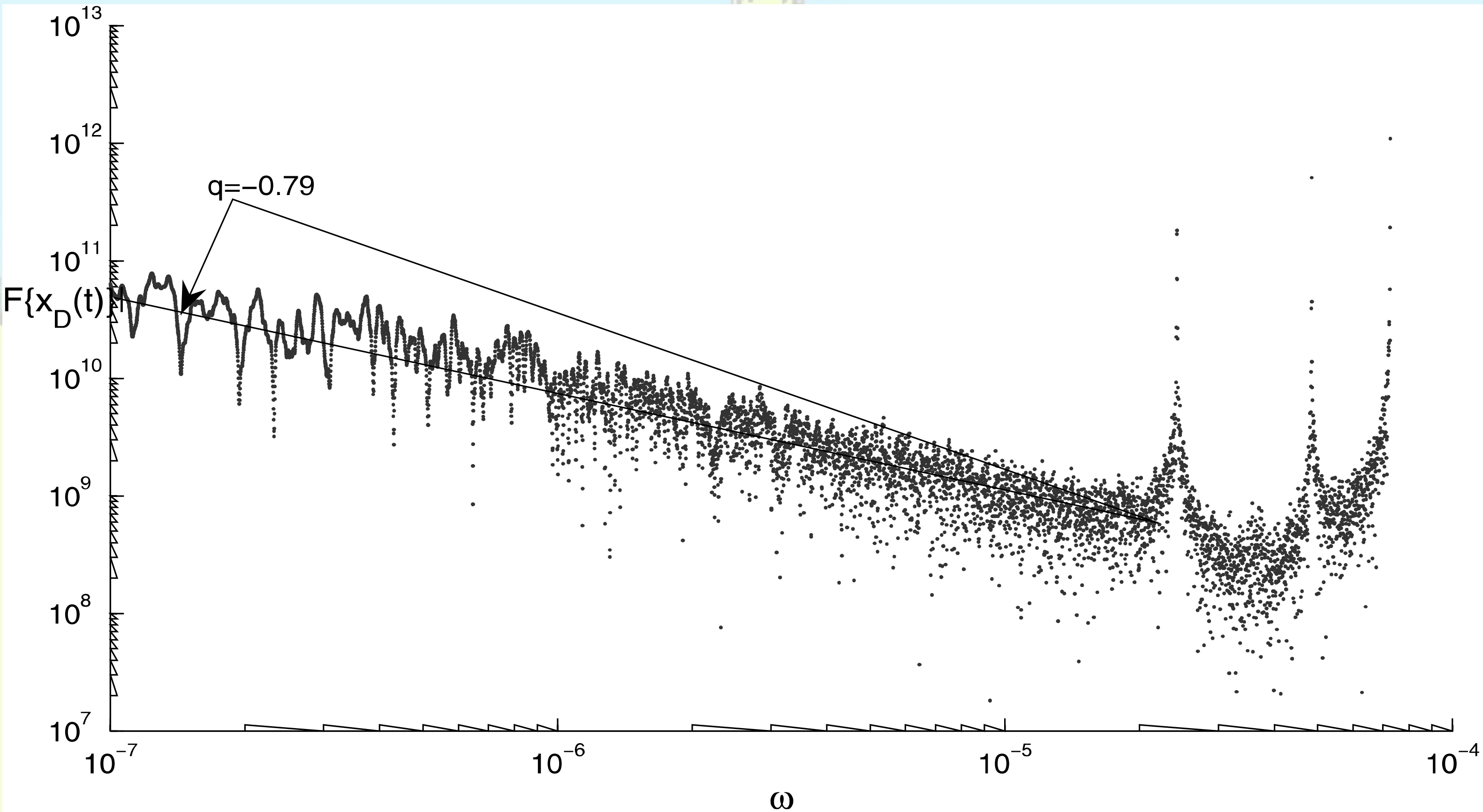


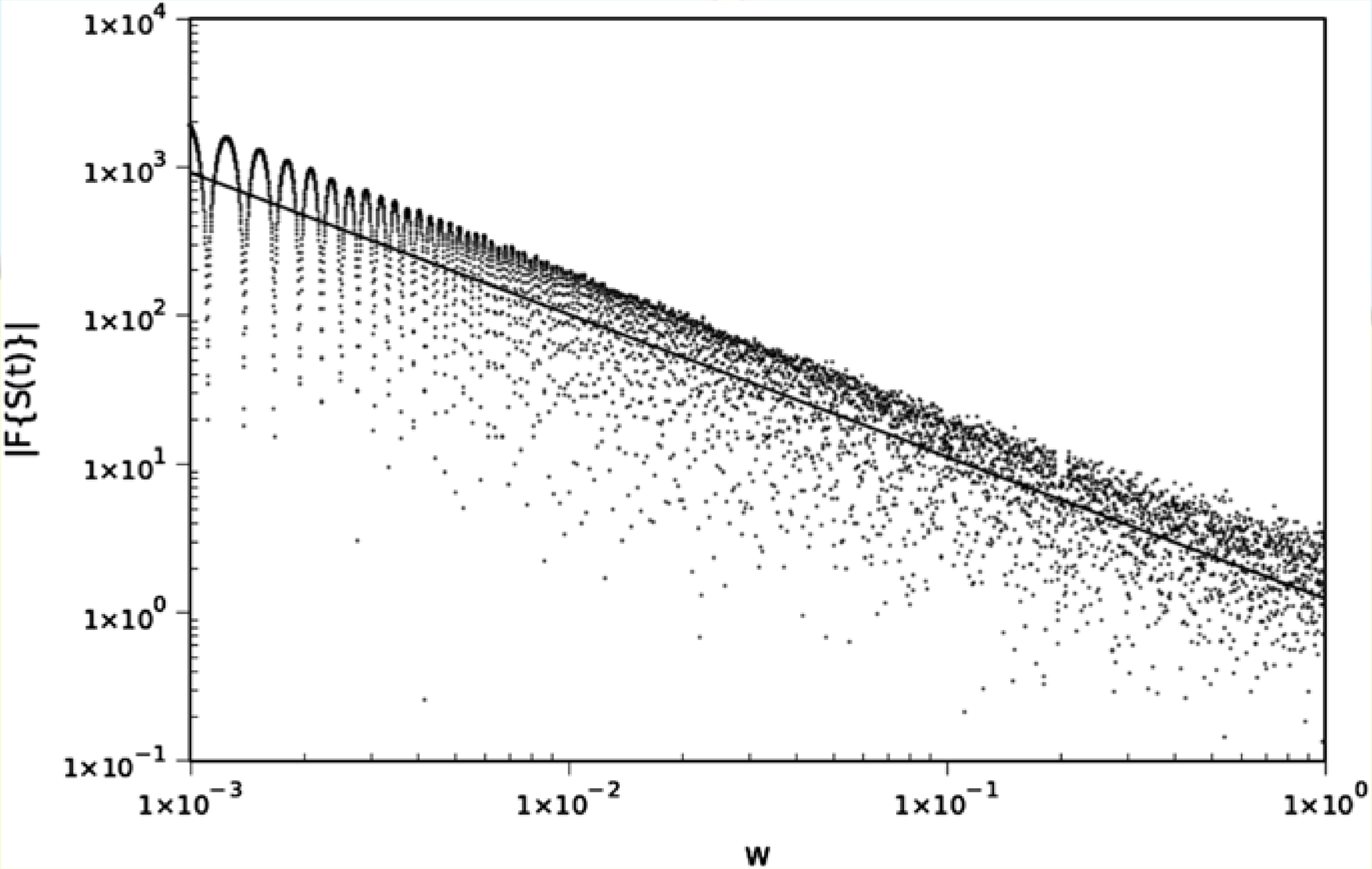



# Spectrum of the monthly average temperatures of Lisbon (1881-2011)

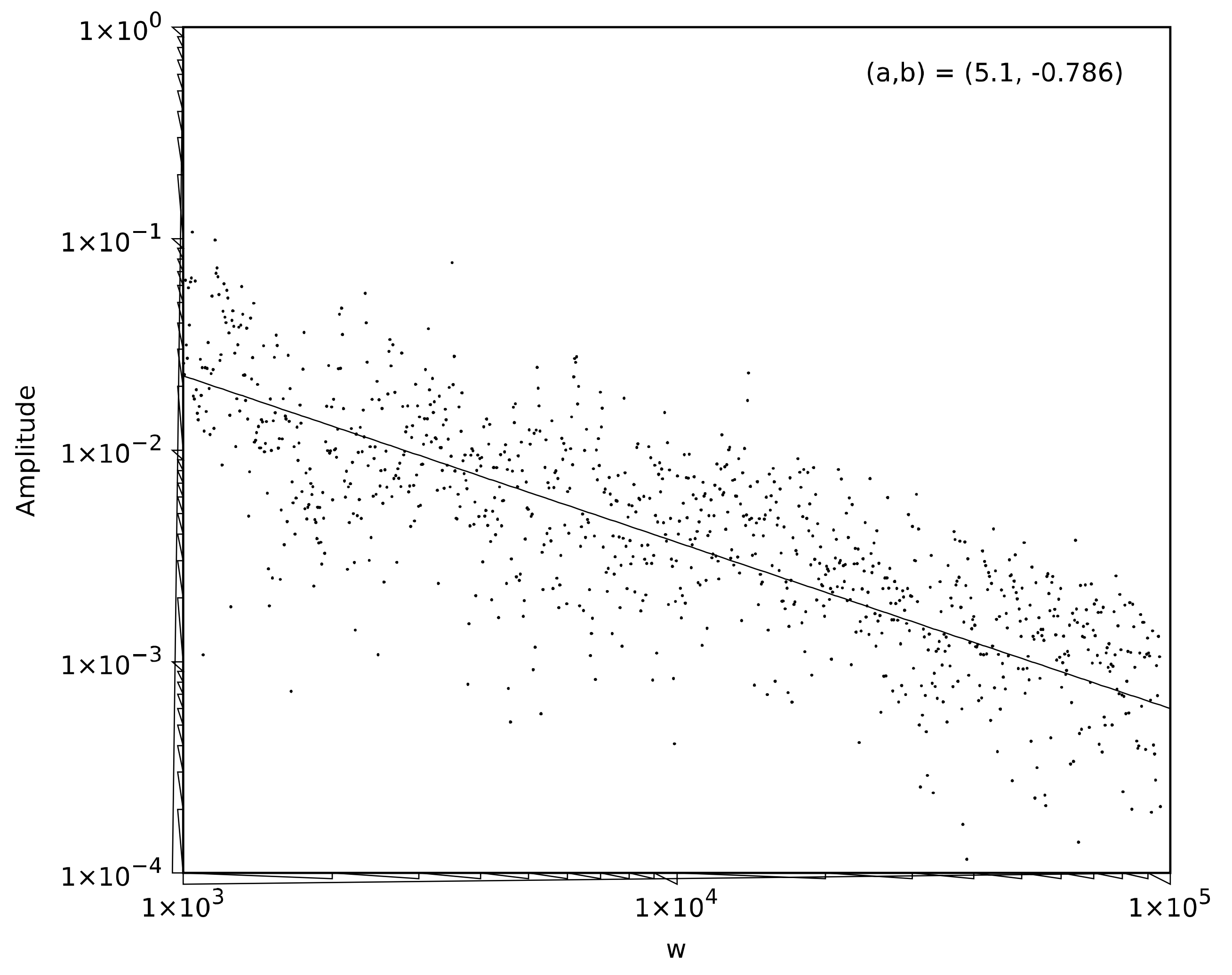



q=−0.79
F{x_D(t)}
10^13
10^12
10^11
10^10
10^9
10^8
10^7
10^−7
10^−6
10^−5
10^−4
ω

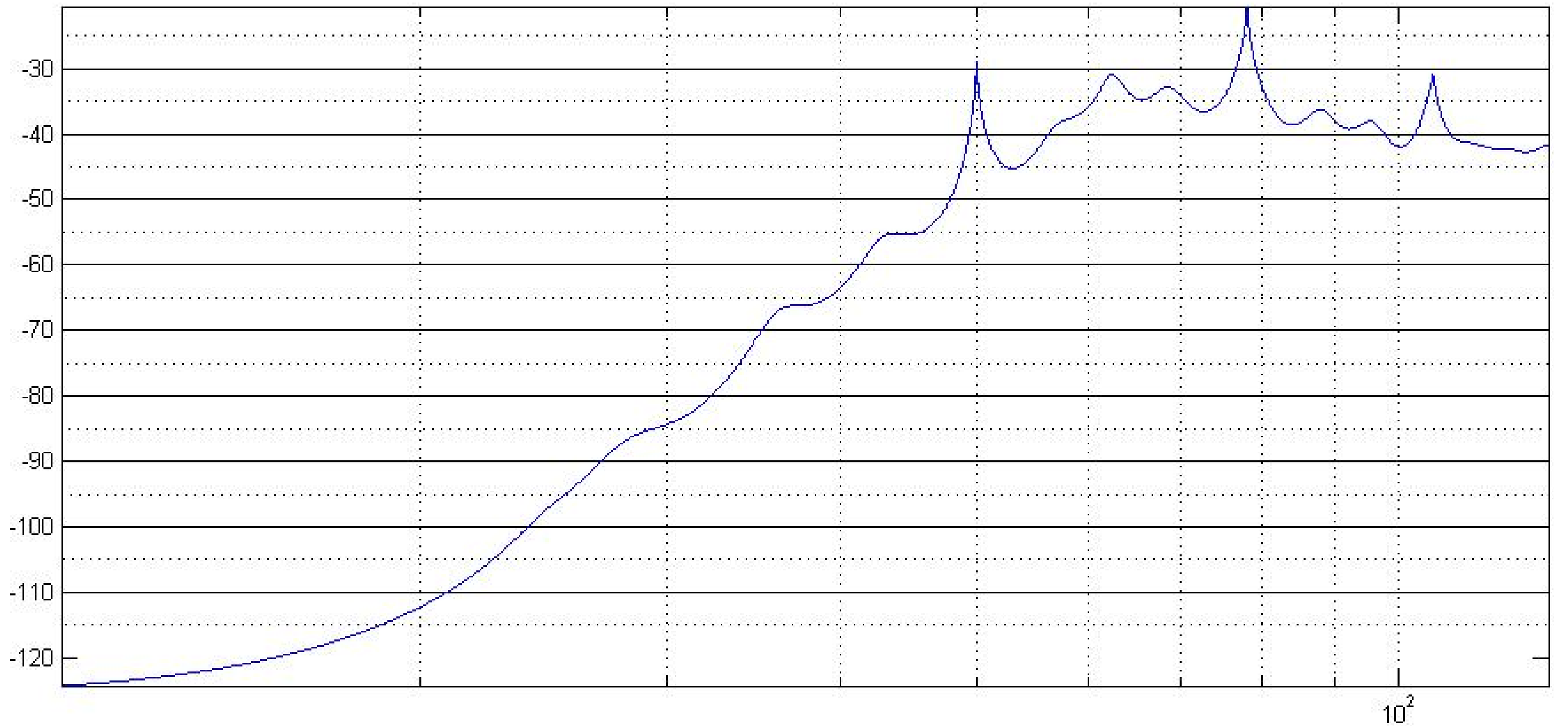
# Fourier transform of the signal for the Human chromosome 1



# Music spectrum



# The spectrum of EEG



# Impedance of vegetables

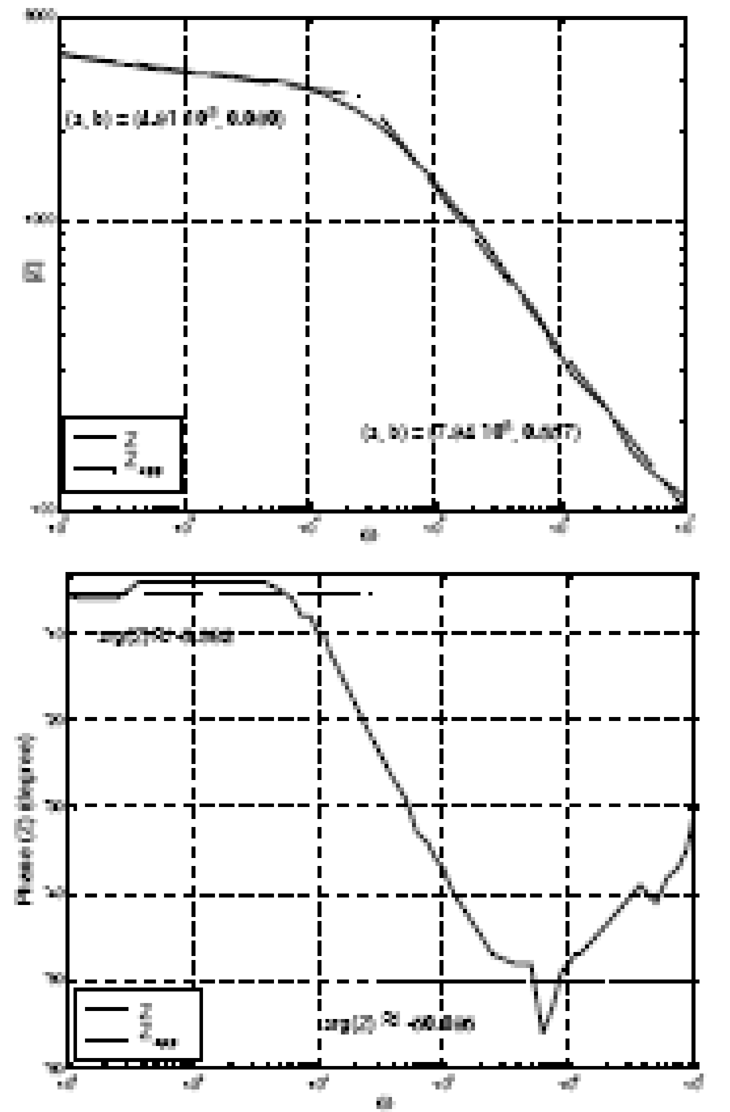
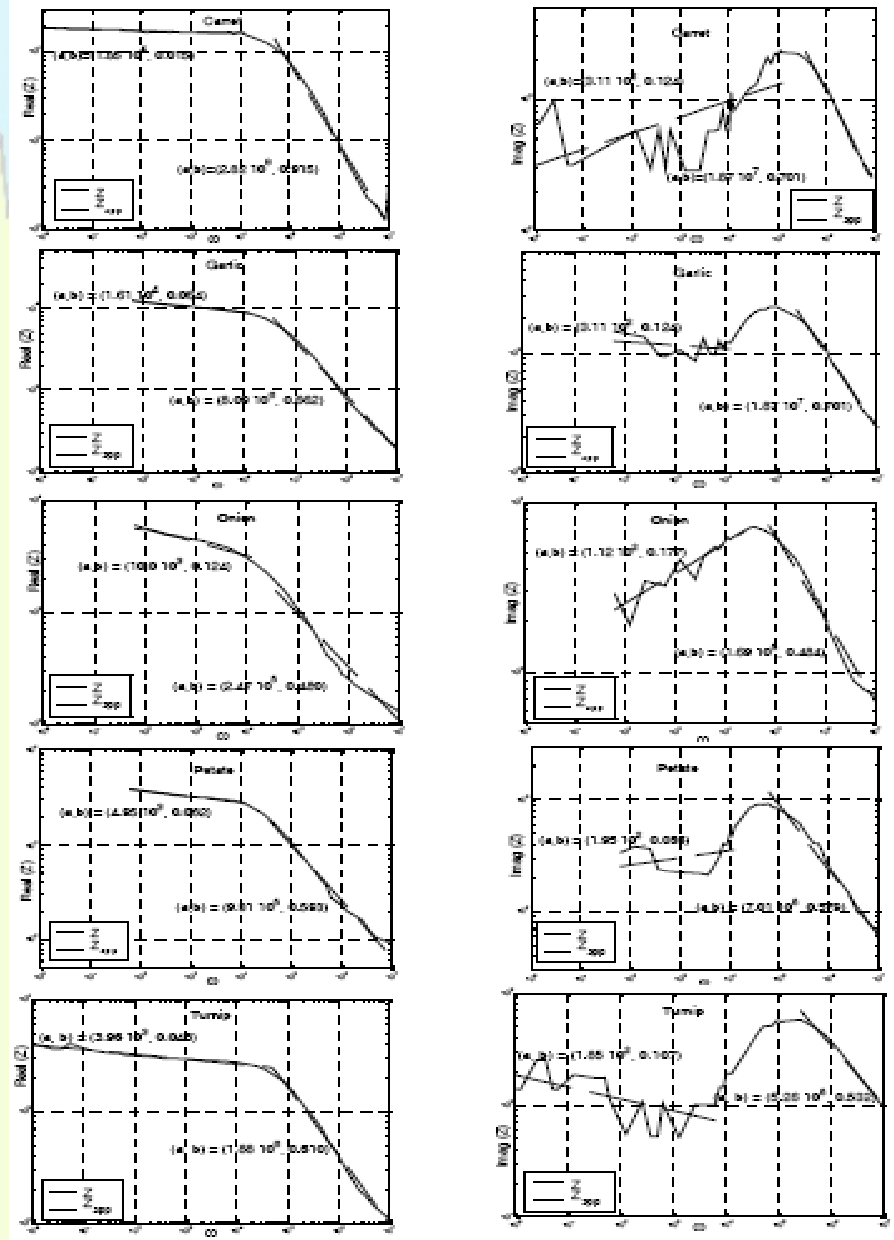
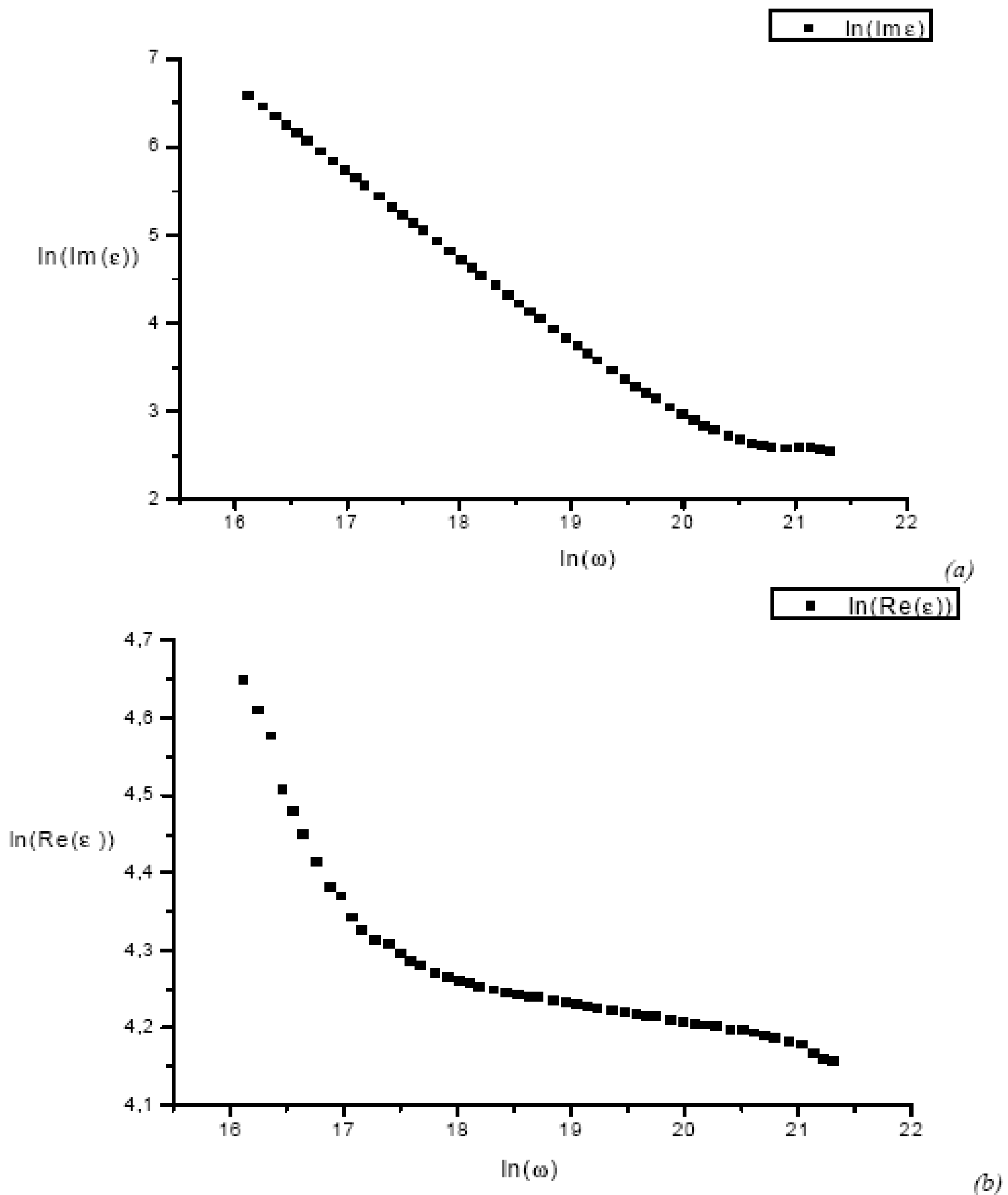


Fig 2. Bode diagrams of the impedance $Z(j\omega)$ for the potato.

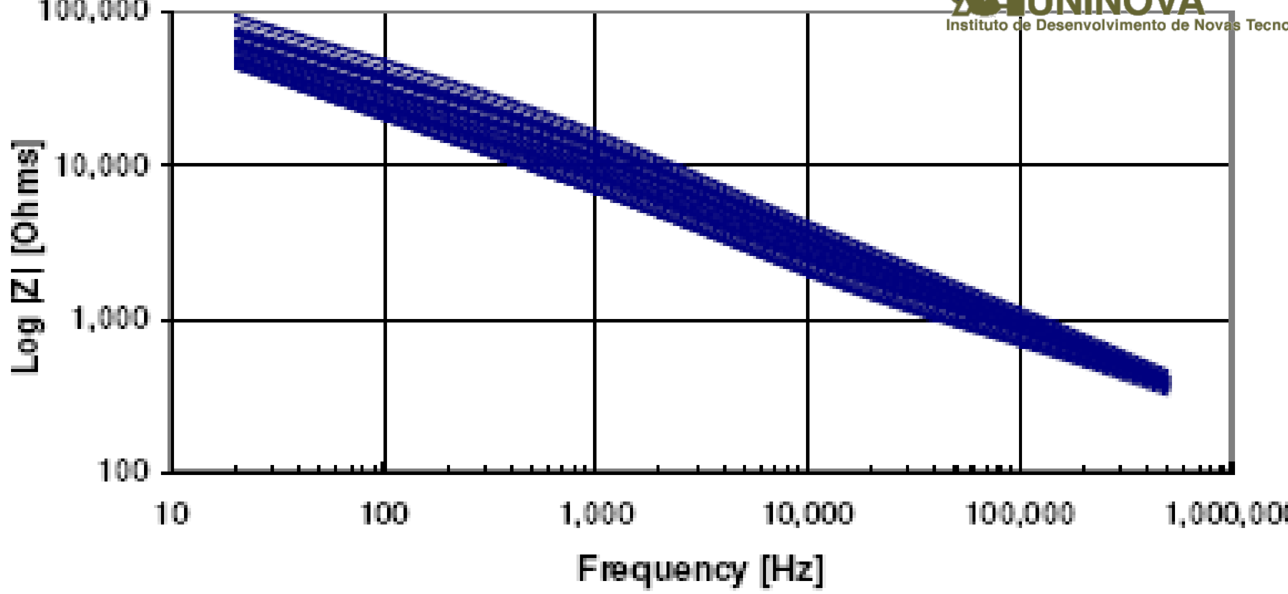


# Permitivity of a melon

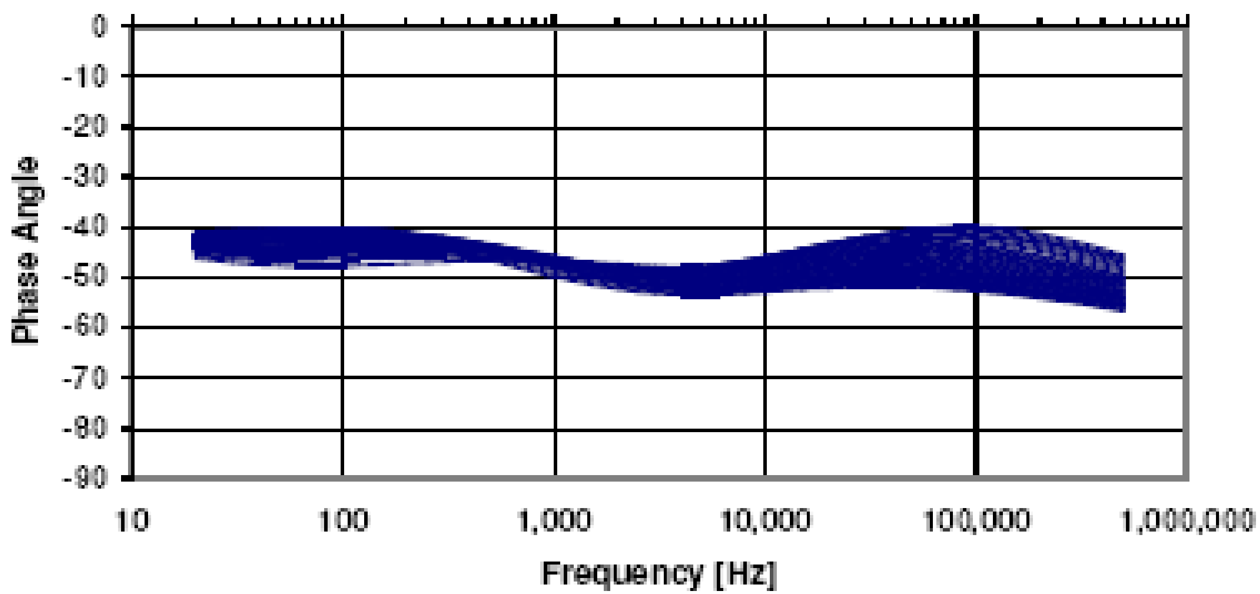


# The fractor

$$Z(s) = \frac{K}{s^{\alpha}}$$

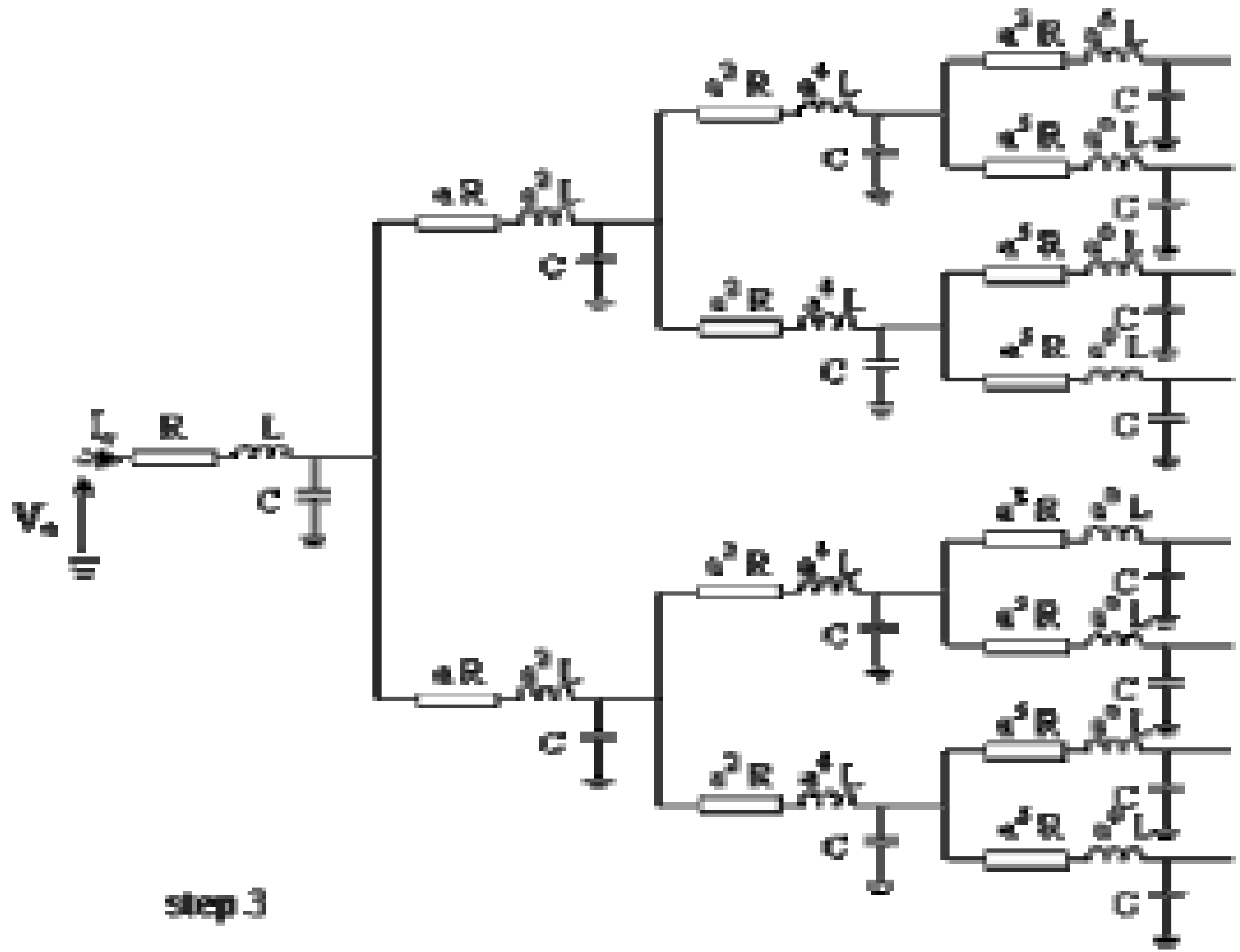



(a)

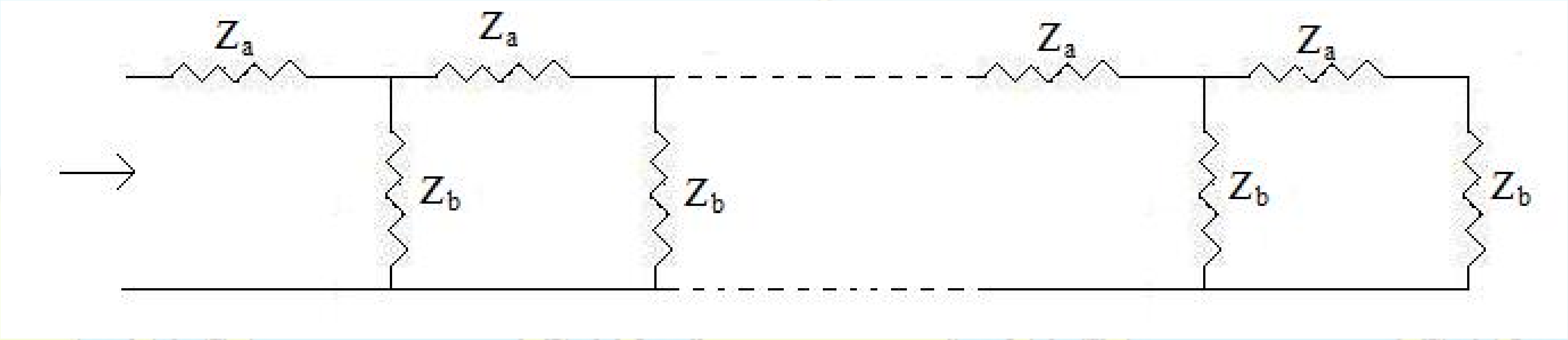



(b)

# Electrical networks



# Infinite Transmission line.

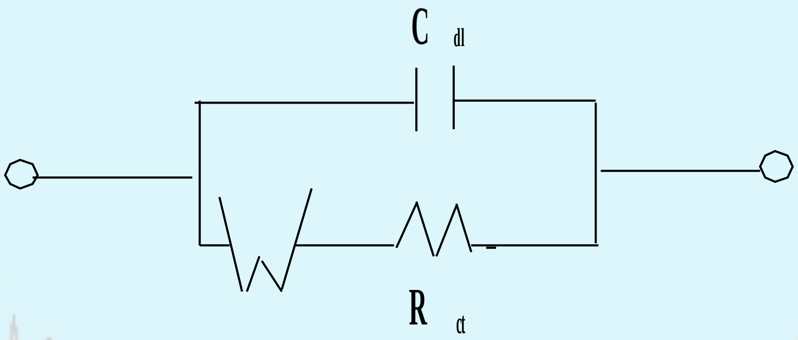



Equivalent impedance $Z = \sqrt{Z_a Z_b}$

when $Z_a = R$ and $Z_b = \frac{1}{SC}$

$$Z = \sqrt{\frac{R}{C}} S^{-1/2}$$ (Fractional order system)

# Warburg Impedance



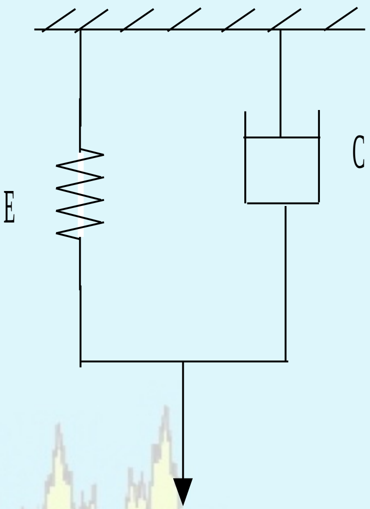

Equivalent circuit of impedance behaviour of a capacitive device immersed in a polarizable medium (e.g. water).

W: Warburg impedance

$$W = Qs^{-1/2}$$ :Half order system.

Diffusion of ions through a porous medium also results in fractional behaviour.

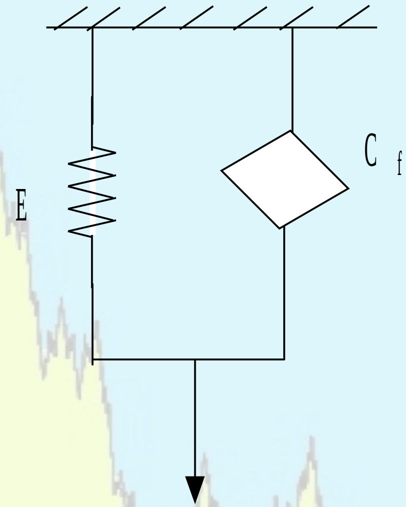
# Viscoelasticity




Kelvin-Voigt model

$$\sigma(t) = \left[ E + C \frac{d}{dt} \right] \varepsilon(t)$$

Integer order model




Fractional Kelvin-Voigt model

$$\sigma(t) = \left[ E + C_f \frac{d^{\alpha}}{dt^{\alpha}} \right] \varepsilon(t)$$

Fractional order model

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# fBm – conventional formulation

$$B_H(t)-{}_H(0)=\frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}\left[(t-\tau)^{H-1/2}-(-\tau)^{H-1/2}\right]w(\tau)d\tau\right\}+$$

$$+\frac{1}{\Gamma(H+1/2)}\left\{+\int_{0}^{t}(t-\tau)^{H-1/2}\,w(\tau)d\tau\right\}$$

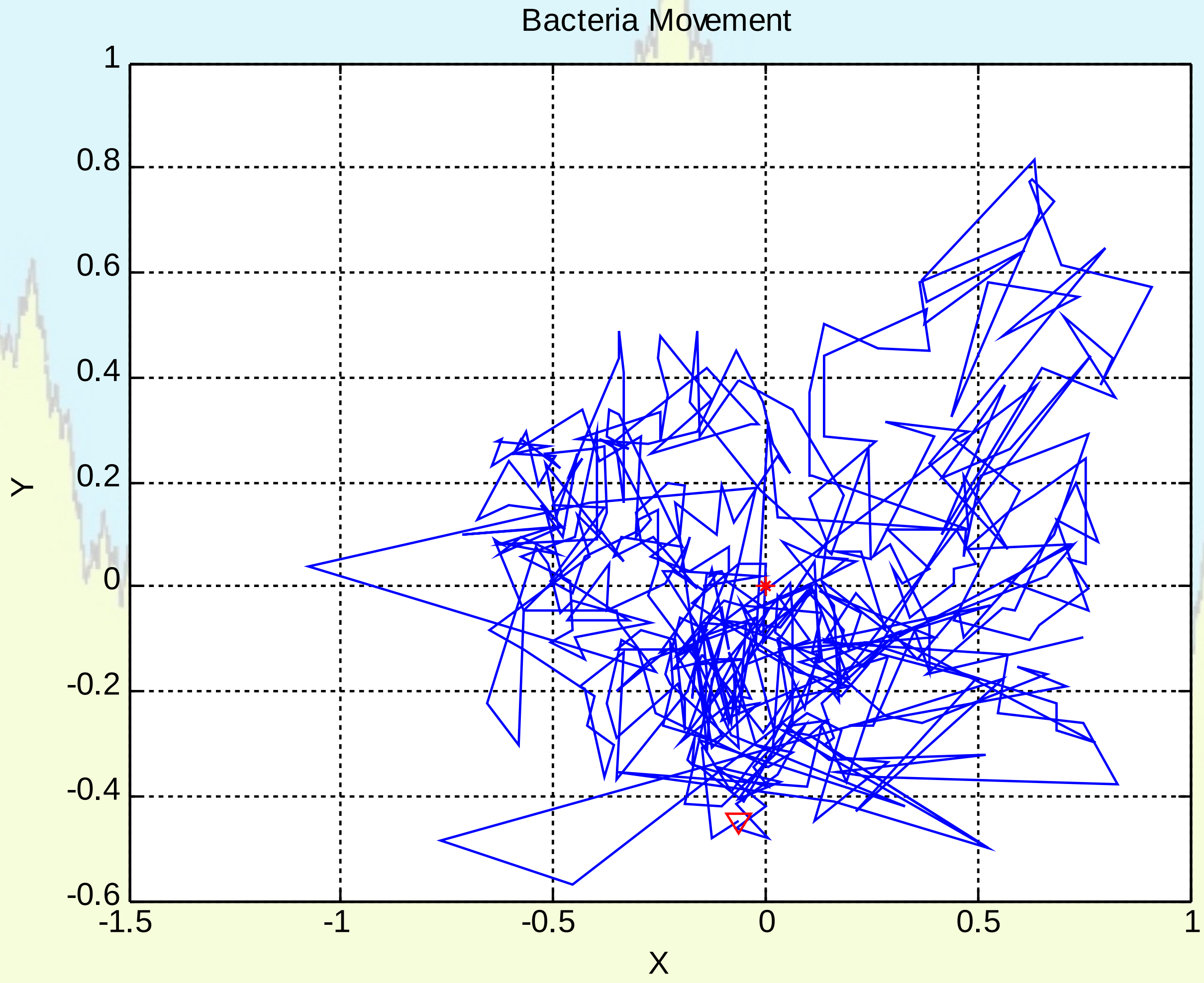
# fBm – general case

$$B_H(t)-B_H(0) = \int_0^t D^{\alpha} w(\tau)\, d\tau$$

For all the fractional derivatives

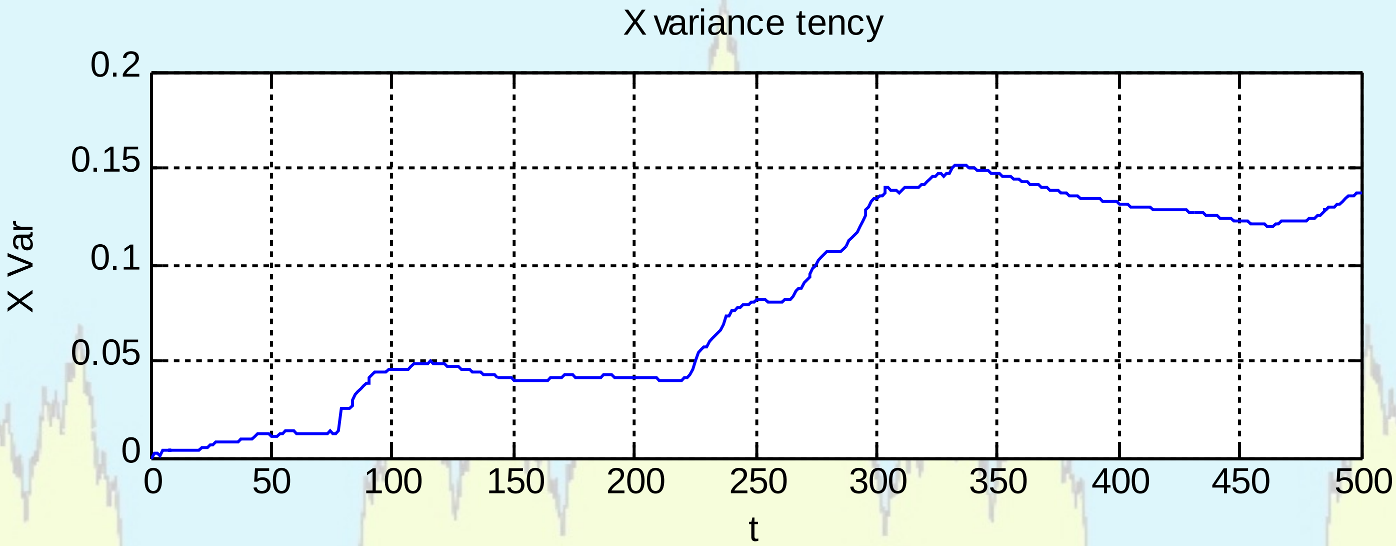
# chemotaxis behavior

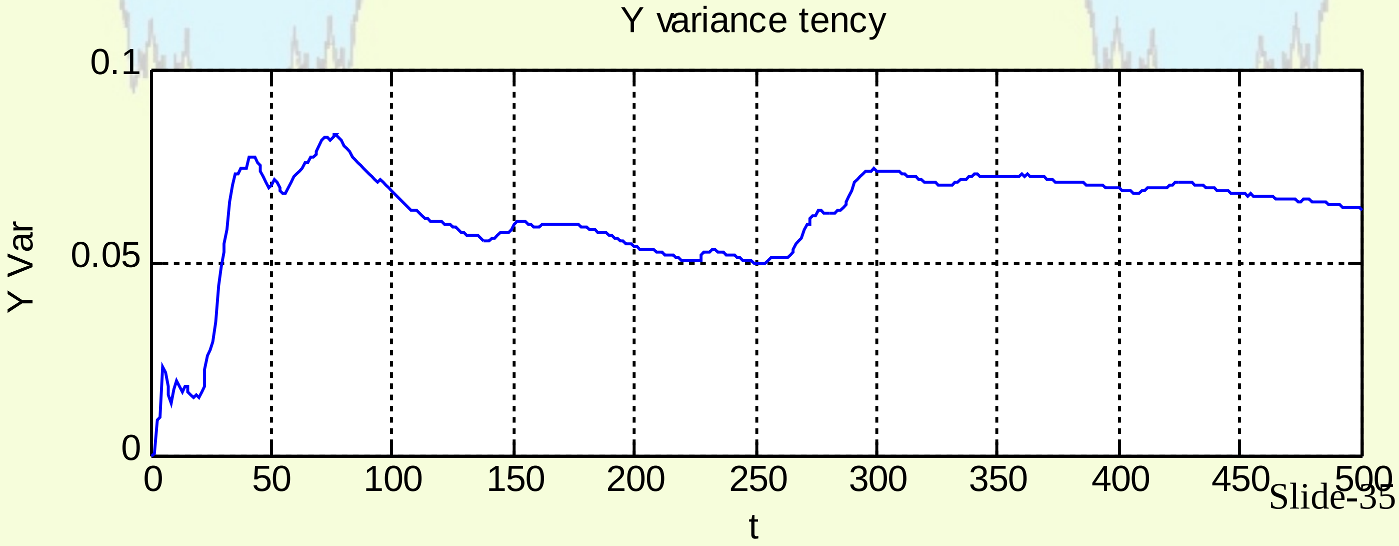
## Sample trajectory of one bacterium



# chemotaxis behavior

## Sample variances



# Returning to the ECG

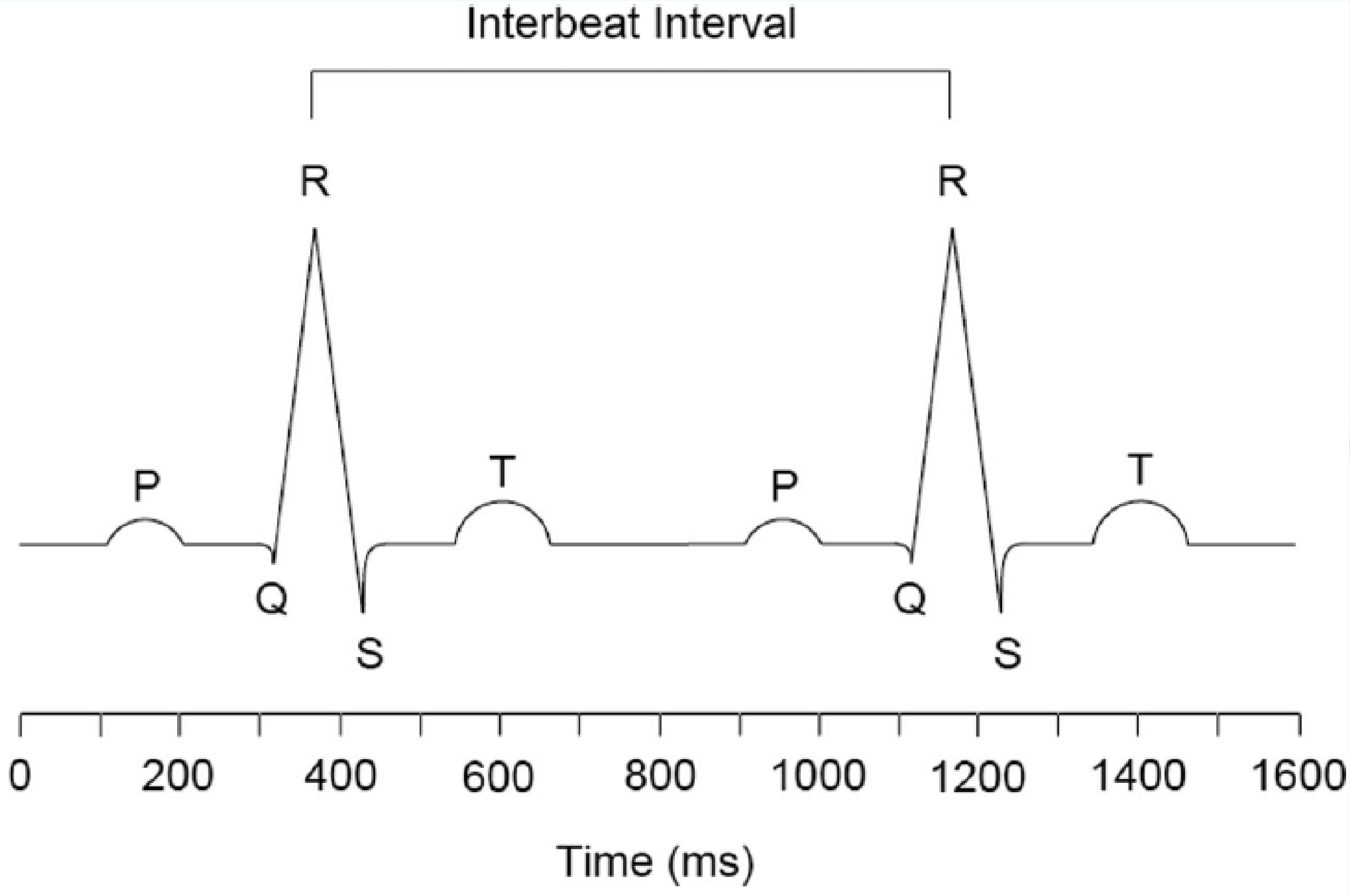



*Figure 1.* Idealized electrocardiograph segment representing two heartbeats. Waveforms are labeled with letters and correspond with specific electrophysiological events during a heartbeat. The interbeat interval is defined by the temporal distance between R-spikes, the waveforms corresponding to depolarization of the heart's ventricles.

# ECG - HRV

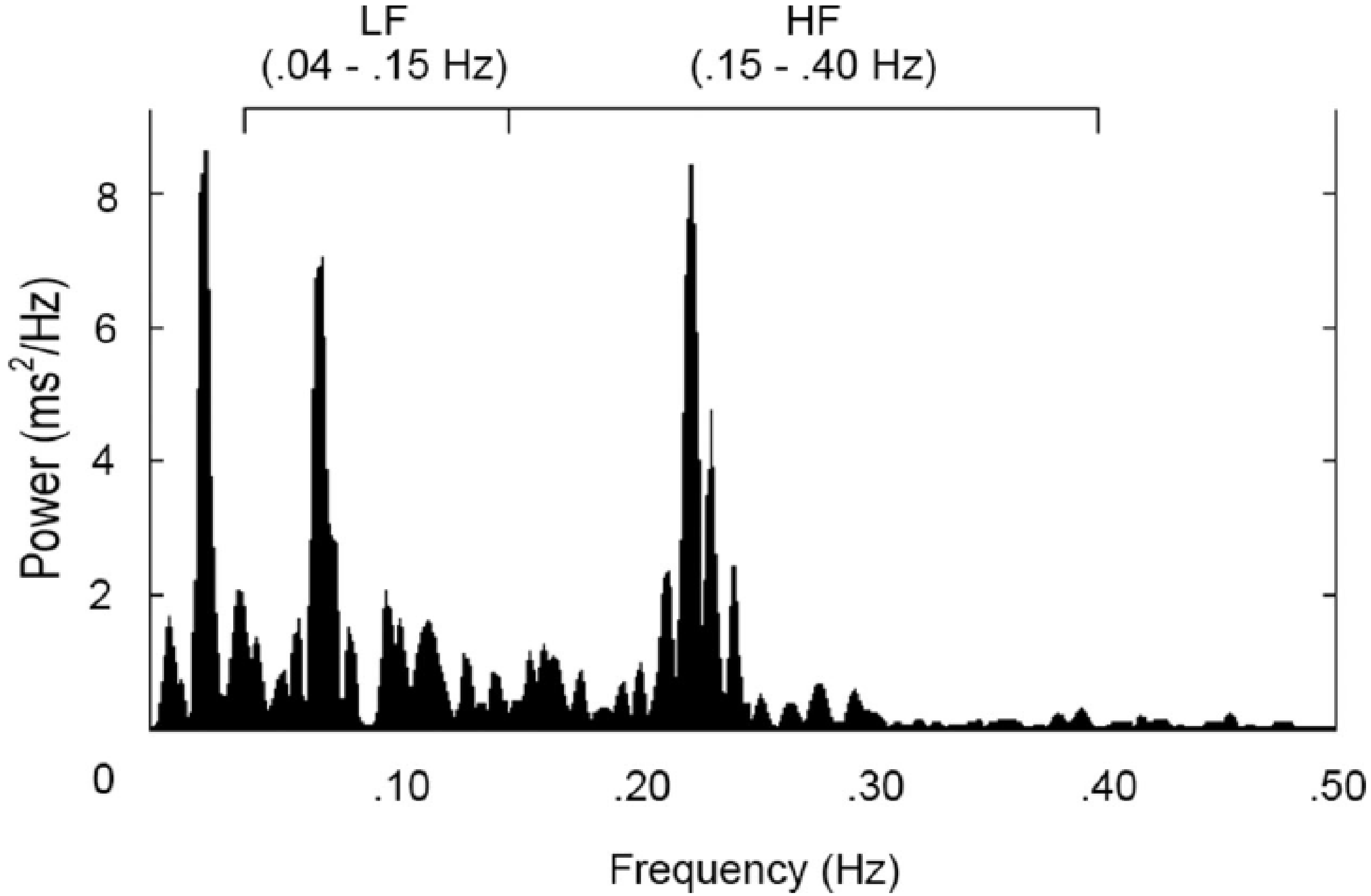
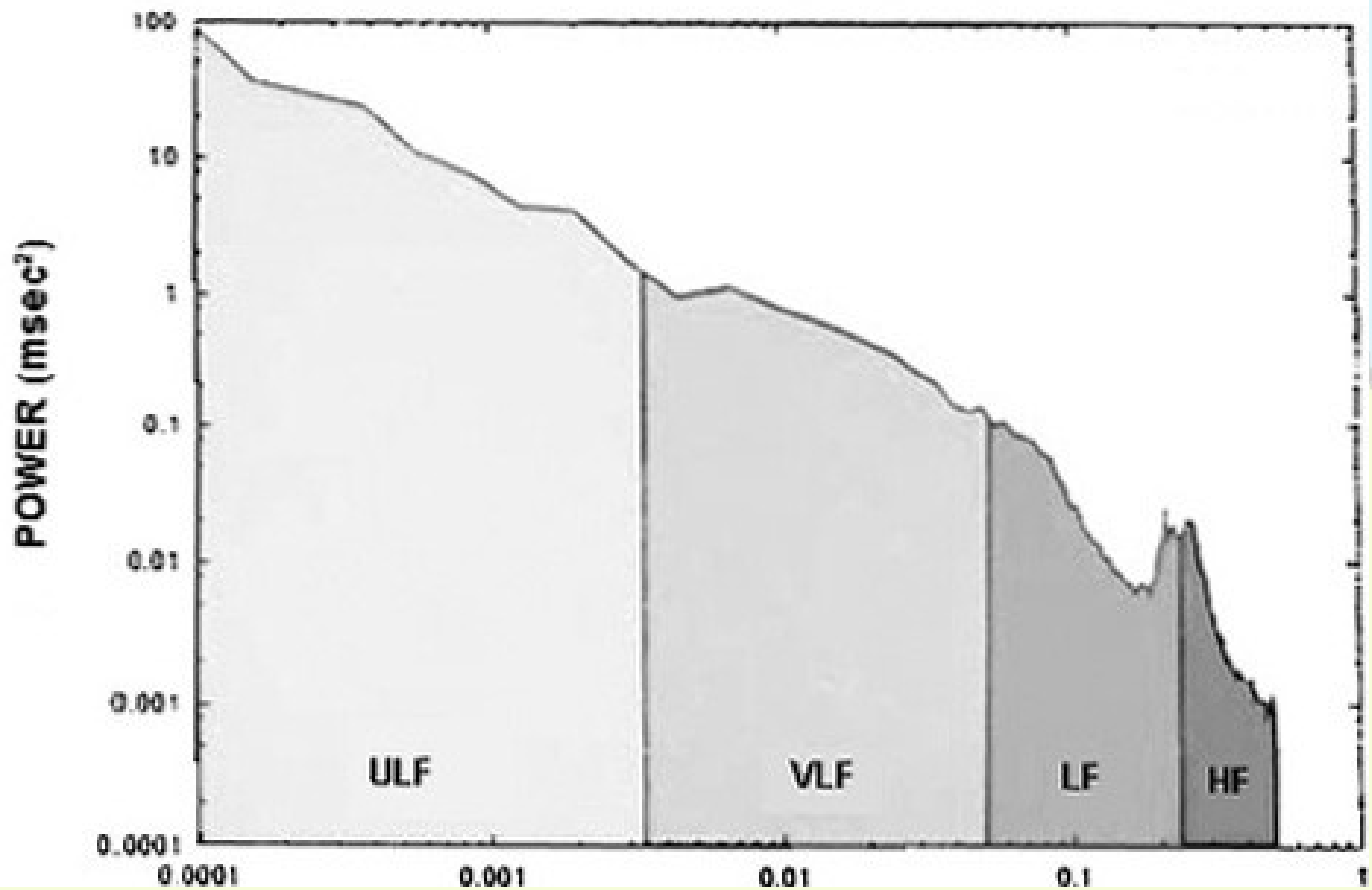



*Figure 2.* An example of a heart rate variability power spectrum obtained using the fast Fourier transform on a 5-min recording obtained from a resting subject in supine position. The low-frequency (LF) component occurs between .04 and .15 Hz and the high-frequency (HF) component occurs between .15 and .40 Hz. Hz = cycles per second.

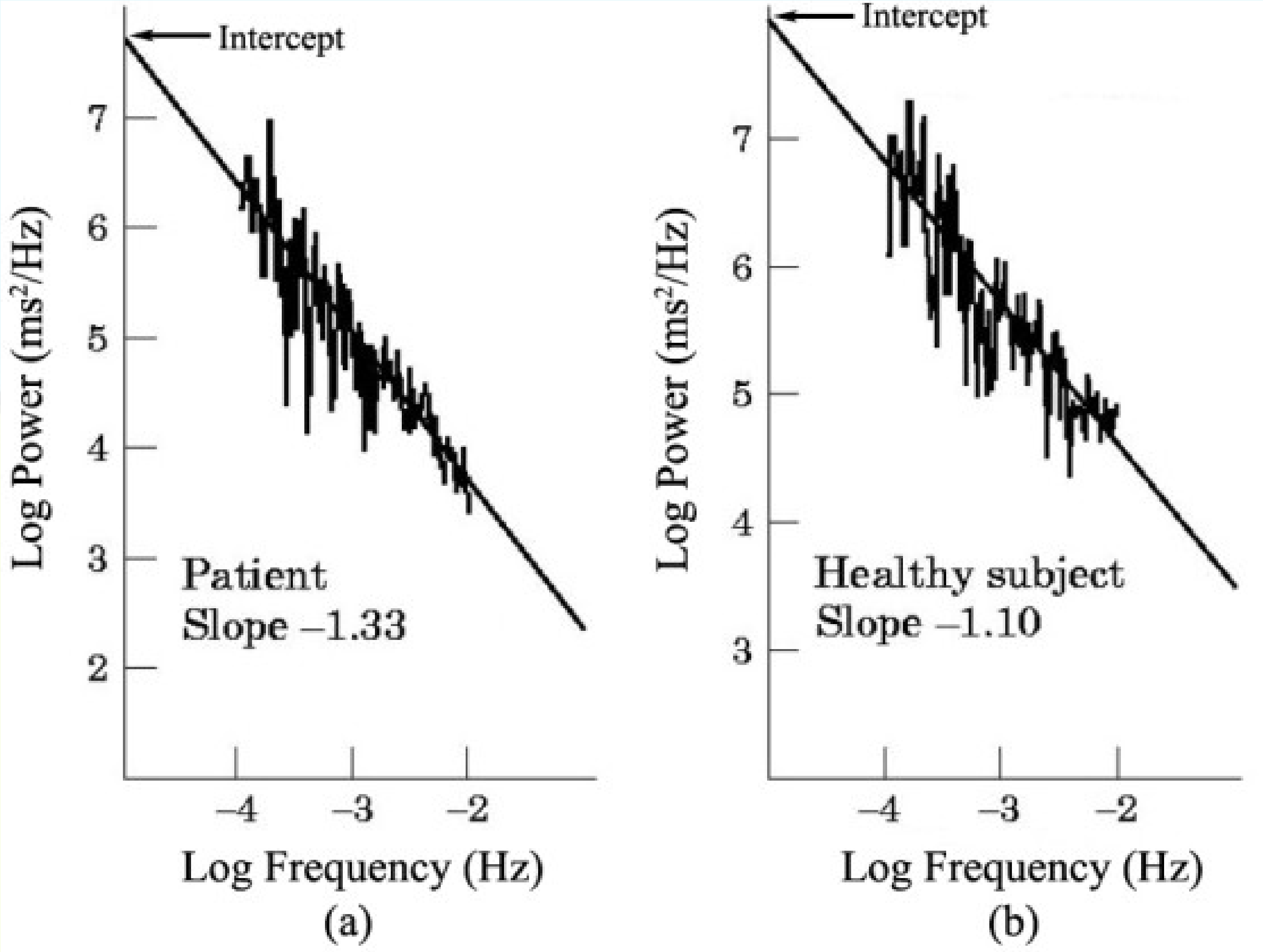
# ECG – HRV spectrum




Log-log plot of the HRV power spectrum over 24 hours. The region between 0.01 and 0.0001 Hz is used to calculate power law slope. (x-axis: frequency Hz)

# ECG – HRV spectrum

short-term fractal scaling exponent (1995)




Examples of the power law slope in a) a patient with cardiac disease. And b) a healthy person.

# The Laplace Transform(s)

**One-sided LT:** $\Rightarrow F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$

$$LT[f^{(\alpha)}(t)] = s^{\alpha} F(s) - \sum_{i=0}^{n-1} [D^{\alpha-1-i} f(0^{+})].s^{i}$$

# The Laplace Transform(s)

**Two-sided LT:** $\Rightarrow F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$

⇩

$LT[f^{(\alpha)}(t)] = s^{\alpha} F(s)$

# Fractional derivatives

**Riemann-Liouville** **Caputo**

**Riesz** **Weyl**

**Hadamard** **Grünwald-Letnikov**

**Marchaud ...**

**ARE THEY EQUIVALENT?**

# Fractional Integral

| | Definition |
|---|---|
| Liouville integral<br>$\alpha>0$ | $D^{-\alpha}\varphi(t)=\frac{1}{(-1)^{\alpha}\Gamma(\alpha)}\int_{0}^{+\infty}\varphi(t+\tau)\tau^{\alpha-1}d\tau$ |
| Riemann integral<br>$\alpha>0$ | $D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau$ |
| Hadamard integral | $D^{-\alpha}(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}\int_{0}^{1}\varphi(t\tau).(1-\tau)^{\alpha-1}d\tau$ |
| Riemann-Liouville integral | $D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}\frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau \quad \alpha>0$ |
| Backward Riemann-Liouville integral | $D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{b}\frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau \quad \alpha>0$ |
| Generalised function<br>(Cauchy) | $D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}\varphi(\tau).(t-\tau)^{\alpha-1}\,d\tau$ |

# Fractional Derivative

| | Definition |
|---|---|
| Left side Riemann-Liouville derivative | $D^{\alpha}\varphi(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \varphi(\tau).(t-\tau)^{\alpha-n-1}d\tau$ t>a |
| Right side Riemann-Liouville derivative | $D^{\alpha}\varphi(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b \varphi(\tau).(\tau-t)^{\alpha-n-1}d\tau$ t<b |
| Left side Caputo derivative | $D^{\alpha}\varphi(t)=\frac{1}{\Gamma(-\nu)}\left[\int_0^t \varphi^{(n)}(\tau).(t-\tau)^{\nu-1}\,d\tau\right]$ t>0 |
| Right side Caputo derivative | $D^{\alpha}\varphi(t)=\frac{1}{\Gamma(-\nu)}\left[\int_t^{+\infty} \varphi^{(n)}(\tau).(\tau-t)^{\nu-1}\,d\tau\right]$ |
| Generalised function (Cauchy) | $D^{\alpha}\varphi(t)=\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^t \varphi(\tau).(t-\tau)^{-\alpha-1}\,d\tau$ |

# Going into the derivative (1)

$$f_{+}^{(1)}(t) = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}$$

$$f_{-}^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$f_{0}^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h/2) - f(t-h/2)}{h}$$

**ARE THEY EQUIVALENT?**

# Going into the derivative (2)

$$f_{+}^{(1)}(t) = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h} \qquad f_{-}^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$s = \lim_{h \to 0} \frac{(1 - e^{-sh})}{h} \qquad = \lim_{h \to 0} \frac{(e^{sh} - 1)}{h}$$

**What happens when |s| goes to infinite?**

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# Going into the derivative (3)

$$f_{+}^{(2)}(t) = \lim_{h\to 0} \frac{f^{(1)}(t) - f^{(1)}(t-h)}{h} =$$

$$= \lim_{h\to 0} \frac{\lim\limits_{h\to 0} \frac{f(t) - f(t-h)}{h} - \lim\limits_{h\to 0} \frac{f(t-h) - f(t-2h)}{h}}{h}$$

$$= \lim_{h\to 0} \frac{\lim\limits_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h}}{h} = \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}$$

# Going into the derivative (4)

$$f_+^{(2)}(t) = \lim_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2} \Rightarrow s^2 = \lim_{h \to 0} \frac{(1 - e^{-sh})^2}{h^2}$$

$$f_-^{(2)}(t) = \lim_{h \to 0} \frac{f(t+2h) - 2f(t-h) + f(t)}{h^2} \Rightarrow s^2 = \lim_{h \to 0^+} \frac{(e^{sh} - 1)^2}{h^2}$$

# Going into the derivative (5)

$$f_{+}^{(N)}(t) = \lim_{h\to 0} \frac{\sum_{k=0}^{N} (-1)^{k} \binom{N}{k} f(t-kh)}{h^{N}} \Rightarrow s^{N} = \lim_{h\to 0} \frac{(1-e^{-sh})^{N}}{h^{N}}$$

$$f_{-}^{(N)}(t) = \lim_{h\to 0} \frac{(-1)^{N} \sum_{k=0}^{N} (-1)^{k} \binom{N}{k} f(t+kh)}{h^{N}} \Rightarrow s^{N} = \lim_{h\to 0} \frac{(e^{sh}-1)^{N}}{h^{N}}$$

They give the $N^{th}$ derivative in ONE step

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# Going into the anti-derivative (1)

$$f_+(t) = \lim_{h\to 0} \frac{f_+^{(-1)}(t) - f_+^{(-1)}(t-h)}{h} \Rightarrow f_+^{(-1)}(t) = \lim_{h\to 0}\left[ hf_+(t) - f_+^{(-1)}(t-h)\right]$$

$$= \lim_{h\to 0}\left[ hf_+(t) + hf_+(t-h) - f_+^{(-1)}(t-2h)\right]$$

$$f_+^{(-1)}(t) = \lim_{h\to 0} h\sum_{k=0}^{\infty} f(t-kh) \Rightarrow \quad s^{-1} = \lim_{h\to 0}\frac{h}{(1-e^{-sh})} \quad \mathrm{Re}(s)>0$$

$$f_-^{(-1)}(t) = \lim_{h\to 0} -h\sum_{k=0}^{\infty} f(t+kh) \Rightarrow \quad s^{-1} = \lim_{h\to 0}\frac{h}{(e^{sh}-1)} \quad \mathrm{Re}(s)<0$$

Essentially the Riemann integral definition!

# Going into the anti-derivative (2)

$$f_{+}^{(-2)}(t) = \lim_{h\to 0} h^2 \sum_{k=0}^{\infty} (k+1) f(t-kh) \Rightarrow s^{-2} = \lim_{h\to 0} \frac{h^2}{(1-e^{-sh})^2} \quad \mathrm{Re}(s) > 0$$

$$f_{-}^{(-2)}(t) = \lim_{h\to 0} h^2 \sum_{k=0}^{\infty} (k+1) f(t+kh) \Rightarrow s^{-2} = \lim_{h\to 0} \frac{h^2}{(e^{sh}-1)^2} \quad \mathrm{Re}(s) < 0$$

The repeated Riemann integral!

# Joining the derivative and anti-derivative transfer functions

$$s^{\pm N} = \lim_{h \to 0} \frac{(1 - e^{-sh})^{\pm N}}{h^{\pm N}} \qquad Re(s) > 0$$

$$s^{\pm N} = \lim_{h \to 0} \frac{(e^{sh} - 1)^{\pm N}}{h^{\pm N}} \qquad Re(s) < 0$$

# Fractionalising the transfer function

$$s^{\alpha} = \lim_{h \to 0^+} \frac{(1 - e^{-sh})^{\alpha}}{h^{\alpha}} = \lim_{h \to 0^+} \frac{(e^{sh} - 1)^{\alpha}}{h^{\alpha}}$$

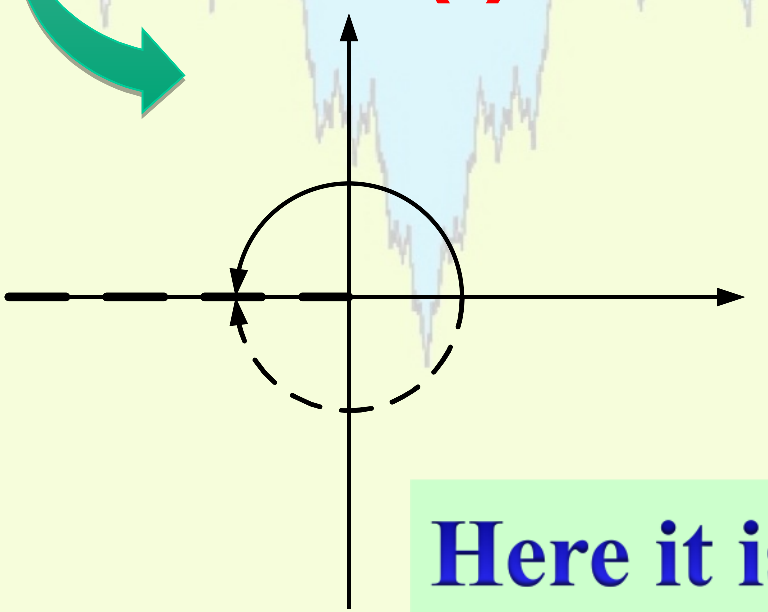
$\mathrm{Re}(s) > 0$ $\qquad$ $\mathrm{Re}(s) < 0$

**We must be careful with the branch cut lines due to the branch point at s=0**

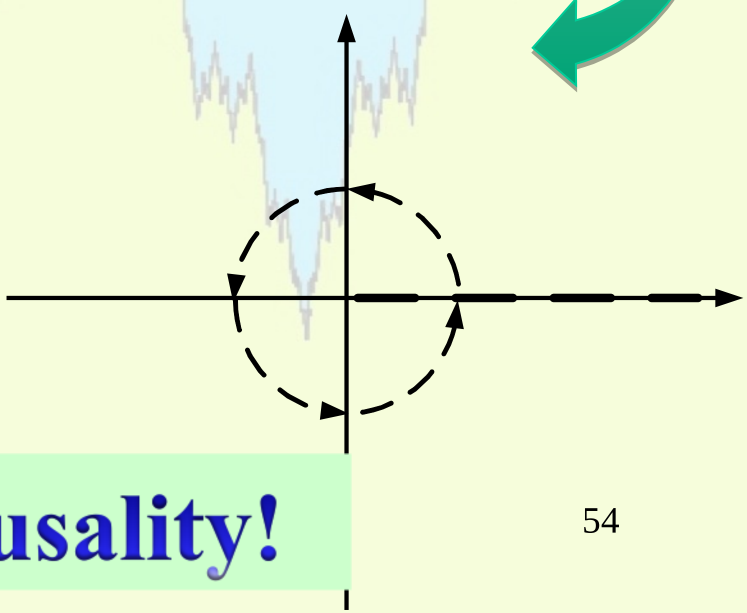
# The differintegrator

$$s^{\alpha} = \lim_{h \to 0^+} \frac{(1 - e^{-sh})^{\alpha}}{h^{\alpha}} = \lim_{h \to 0^+} \frac{(e^{sh} - 1)^{\alpha}}{h^{\alpha}}$$

$Re(s) > 0$



$Re(s) < 0$



**Here it is the causality!**

# Generalisation of a well known property of the Laplace transform

$$[D_f^{\alpha} f(t)] = s^{\alpha} F(s) \quad \text{for } \mathrm{Re}(s) > 0 \quad \text{Forward}$$

$$[D_b^{\alpha} f(t)] = s^{\alpha} F(s) \quad \text{for } \mathrm{Re}(s) < 0 \quad \text{Backward}$$

**There is a system – the differintegrator - that has $s^{\alpha}$ as transfer function.**

# Fractional Differintegrator

- Inverse LT of $s^\alpha$ for Real orders:
  - Causal

$$LT^{-1}[s^\alpha] = \frac{t^{-\alpha-1}}{(\alpha-1)!}u(t)$$

  - Anti-causal

$$LT^{-1}[s^\alpha] = -\frac{t^{-\alpha-1}}{(\alpha-1)!}u(-t)$$

# Liouville differintegration

- Causal

$$x_f^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} x(\tau).(t-\tau)^{-\alpha-1} \, d\tau$$

- Anti-causal

$$x_b^{(\alpha)}(t) = -\frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} x(\tau).(t-\tau)^{-\alpha-1} \, d\tau$$

# Grünwald-Letnikov fractional derivative

$$s^{\alpha} = \lim_{h\to 0^+} \frac{(1 - e^{-sh})^{\alpha}}{h^{\alpha}} = \lim_{h\to 0^+} \frac{(e^{sh} - 1)^{\alpha}}{h^{\alpha}}$$

Re(s) > 0 Re(s) < 0

$$f_f^{(\alpha)}(t) = \lim_{h\to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^{\alpha}}$$

forward

backward

$$f_b^{(\alpha)}(t) = \lim_{h\to 0^+} e^{-j\alpha\pi} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t+kh)}{h^{\alpha}}$$

# Derivative of the exponential

If $f(t)=e^{st}$

$$f_f^{(\alpha)}(t)=e^{at}\lim_{h\to 0^+}\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}e^{-kh}}{h^{\alpha}}=e^{at}\lim_{h\to 0^+}\frac{(1-e^{-ah})^{\alpha}}{h^{\alpha}}=s^{\alpha}\,e^{st}\quad \text{if } \operatorname{Re}(s)>0$$

$$f_b^{(\alpha)}(t)=e^{at}\lim_{h\to 0^+}(-1)^{\alpha}\frac{\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}e^{kh}}{h^{\alpha}}=e^{at}\lim_{h\to 0^+}\frac{(e^{ah}-1)^{\alpha}}{h^{\alpha}}=s^{\alpha}\,e^{st}\quad \text{if } \operatorname{Re}(s)<0$$

# Forward derivative of the sinusoid

$f(t) = e^{j\omega t} \quad \omega > 0 \Rightarrow f_f^{(\alpha)}(t) = (j\omega)^{\alpha} e^{j\omega t}$

Then

$D^{\alpha}\cos(\omega t) =$

$= D^{\alpha}[e^{j\omega t} + e^{-j\omega t}]/2 = 1/2(j\omega)^{\alpha} e^{j\omega t} + 1/2(-j\omega)^{\alpha} e^{-j\omega t} =$

$= \omega^{\alpha} \cos(\omega t + \alpha\pi/2)$

and, similarly

$D^{\alpha}\sin(\omega t) = \omega^{\alpha} \sin(\omega t + \alpha\pi/2)$

**What about the backward?**

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# Going into the derivative (1)

$$f_0^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h/2) - f(t-h/2)}{h}$$

LESS USED

# Going into the derivative (2)

$$f_0^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h/2) - f(t-h/2)}{h}$$

$$\Downarrow$$

$$s = \lim_{h \to 0} \frac{(e^{sh} - e^{-sh})}{h}$$

# Going into the derivative (3)

$$f_0^{(2)}(t) = \lim_{h \to 0} \frac{f^{(1)}(t+h/2) - f^{(1)}(t-h/2)}{h} =$$

$$= \lim_{h \to 0} \frac{\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} - \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}}{h}$$

$$= \lim_{h \to 0} \frac{\lim_{h \to 0} \frac{f(t+h) - 2f(t) + f(t-h)}{h}}{h} = \lim_{h \to 0} \frac{f(t+h) - 2f(t) + f(t-h)}{h^2}$$

# Going into the derivative (4)

$$f_0^{(2)}(t) = \lim_{h \to 0} \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} \Rightarrow s^2 = \lim_{h \to 0} \frac{(e^{sh} - e^{-sh})^2}{h^2}$$

# Going into the derivative (5)

$$f_0^{(N)}(t) = \lim_{h\to 0} \frac{(-1)^{N/2} \sum_{k=-N/2}^{N/2} (-1)^k \frac{\Gamma(N+1)}{\Gamma(N/2+k+1)\,\Gamma(N/2-k+1)} f(t-kh)}{h^N} \quad \text{even } N$$

$$= \lim_{h\to 0} \frac{(-1)^{(N+1)/2} \sum_{k=-(N-1)/2}^{(N+1)/2} (-1)^k \frac{\Gamma(N+1)}{\Gamma((N+1)/2-k+1)\,\Gamma((N-1)/2+k+1)} f(t-kh+h/2)}{h^N}$$

odd N

$$\Rightarrow \quad s^N = \lim_{h\to 0} \frac{(e^{sh} - e^{-sh})^N}{h^N}$$

# Fractionalising the transfer function

$$s^{\alpha} = \lim_{h \to 0} \frac{(e^{sh} - e^{-sh})^{\alpha}}{h^{\alpha}}$$

**Which is the region of convergence?**

# Central Derivatives

- Type 1

$$D_{c_1}^{\alpha} f(t) = \lim_{h \to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha/2-k+1)\,\Gamma(\alpha/2+k+1)} f(t-kh)$$

- Type 2

$$D_{c_2}^{\alpha} f(t) = \lim_{h \to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma[(\alpha+1)/2-k+1]\,\Gamma[(\alpha-1)/2+k+1]} f(t-kh+h/2)$$

# Riesz Potentials

$$\lim_{h\to 0}\frac{\Gamma(\alpha+1)}{h^{\alpha}}\sum_{-\infty}^{+\infty}\frac{(-1)^k}{\Gamma[(\alpha+1)/2-k+1]\,\Gamma[(\alpha-1)/2+k+1]}f(t-kh+h/2)=$$

$$=\frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}f(z-x)\frac{1}{|x|^{\alpha+1}}dx$$

$$\lim_{h\to 0}\frac{\Gamma(\alpha+1)}{h^{\alpha}}\sum_{-\infty}^{+\infty}\frac{(-1)^k}{\Gamma[(\alpha+1)/2-k+1]\,\Gamma[(\alpha-1)/2+k+1]}f(t-kh+h/2)=$$

$$=-\frac{1}{2\Gamma(-\alpha)\sin(\alpha\pi/2)}\int_{-\infty}^{\infty}f(z-x)\frac{\mathrm{sgn}(x)}{|x|^{\alpha+1}}dx$$

# Main areas for research

1) Fractional control of engineering systems,
2) Fundamental explorations of the mechanical, electrical, and thermal constitutive relations and other properties of various engineering materials such as viscoelastic polymers, foam, gel, and animal tissues, and their engineering and scientific applications,
3) Advancement of Calculus of Variations and Optimal Control to fractional dynamic systems,
4) Fundamental understanding of wave and diffusion phenomenon, their measurements and verifications,
5) Analytical and numerical tools and techniques,
6) Bioengineering and biomedical applications,
7) Thermal modeling of engineering systems such as brakes and machine tools,
8) Image and signal processing.

# Where do we go to?

- **EVERYWHERE**

**Fractional Calculus:**

**the Calculus for the XXI$^{th}$ century**

**(Nishimoto)**

**Fractional Systems:**

**The XXI$^{th}$ Century Systems**

**(mdo)**

- **The International Conference on Fractional Signals and Systems 2013**

**October 2013**

**Ghent, Belgium**

**http://www.fss13.ugent.be/**