Superdiffusion and epidemiological spreading

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CMAF, Lisbon, 15th February, 2013

The spatial SIS model

$$S_i + I_j \xrightarrow{\beta} I_i + I_j$$
$$I_i \xrightarrow{\alpha} S_i$$

Master equation

$$
\begin{aligned}
\frac{d}{dt}\, p(I_1, I_2, \dots, I_N, t) &= \sum_{i=1}^{N} \beta \left(\sum_{j=1}^{N} J_{ij} I_j \right) I_i \; p(I_1, \dots, 1 - I_i, \dots, t) \\
&+ \sum_{i=1}^{N} \alpha (1 - I_i) \;\; p(I_1, \dots, 1 - I_i, \dots, I_N, t)
\end{aligned}
$$

$$
- \sum_{i=1}^{N} \left[\beta \left(\sum_{j=1}^{N} J_{ij} I_j \right) (1 - I_i) + \alpha I_i \right] p(I_1, \dots, I_i, \dots, I_N, t)
$$

$$\frac{d}{dt}\langle I_i\rangle = \beta\sum_{j=1}^{N} J_{ij}\langle I_j(1-I_i)\rangle - \alpha\langle I_i\rangle$$

$$= \beta\sum_{j=1}^{N} J_{ij}\langle S_i I_j\rangle - \alpha\langle I_i\rangle$$

An individual based random walk model

$$
\begin{aligned}
S_i + I_j &\xrightarrow{\chi} I_i + S_j \\
I_i + S_j &\xrightarrow{\chi} S_i + I_j
\end{aligned}
$$

The dynamics of the expectation values

$$\frac{d}{dt}\langle I_i\rangle = \chi\sum_{j=1}^{N} J_{ij}\langle S_i I_j\rangle - \chi\sum_{j=1}^{N} J_{ij}\langle I_i S_j\rangle$$

$$= \chi\sum_{j=1}^{N} J_{ij}\langle S_i I_j - I_i S_j\rangle$$

and by inserting $S_i = 1 - I_i$ etc. we obtain
$\langle S_i I_j - I_i S_j\rangle = \langle(1 - I_i)I_j - I_i(1 - I_j)\rangle = \langle I_j\rangle - \langle I_i\rangle$.

$$\frac{d}{dt} \langle I_i \rangle = \chi \sum_{j=1}^{N} J_{ij} \left(\langle I_j \rangle - \langle I_i \rangle \right) \tag{1}$$

with

$$\sum_{j=1}^{N} J_{ij} \left(\langle I_j \rangle - \langle I_i \rangle \right) =: \Delta \langle I_i \rangle \tag{2}$$

$$\sum_{j=1}^{N} J_{ij} \left(\langle I_j \rangle - \langle I_i \rangle \right) = \langle I_{i-1} \rangle - 2 \langle I_i \rangle + \langle I_{i+1} \rangle \quad . \tag{3}$$

If we denote

$$u(\underline{x}_i, t) := \langle I_i \rangle \tag{4}$$

where $\underline{x}_i$ is the location of individual i in the case of regular lattices encoded in the adjacency matrix J then we have

$$\langle I_{i-1} \rangle - 2\langle I_i \rangle + \langle I_{i+1} \rangle \sim \frac{\partial^2}{\partial x^2} u(\underline{x}, t) \quad . \tag{5}$$

and our result

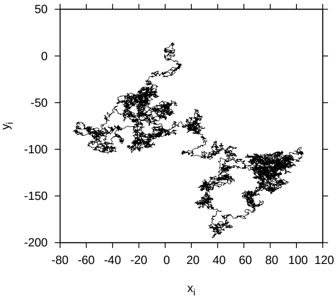
$$\frac{d}{dt} \langle I_i \rangle = \chi \Delta \langle I_i \rangle \tag{6}$$

$$\frac{\partial}{\partial t} u(\underline{x}, t) = \chi \Delta u(\underline{x}, t) \tag{7}$$

describing the random walk of our item being exchanged between individuals i living at locations $\underline{x}_i$.

$$\frac{\partial}{\partial t} u(x,y,t) = \chi \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x,y,t) \qquad (8)$$

Ordinary diffusion



$$\frac{\partial}{\partial t} u(x,t) = \chi \frac{\partial^2}{\partial x^2} u(x,t), \quad \chi > 0 \tag{9}$$

Diffusion equation in Fourier space

$$\frac{\partial}{\partial t}\tilde{u}(k,t) = -k^2\chi\tilde{u}(k,t), \tag{10}$$

where

$$\tilde{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\mathrm{d}x$$

$$\tilde{u}(k,t) = \tilde{u}(k,t_0)e^{-k^2\chi(t-t_0)}. \tag{11}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k,t) e^{ikx} \mathrm{d}k$$

$$u(x,t) = \frac{1}{2\sqrt{\pi\chi(t-t_0)}} \int_{-\infty}^{\infty} u(y,t_0) \mathrm{e}^{-\frac{(x-y)^2}{4\chi(t-t_0)}} \mathrm{d}y.$$

Discrete case

Let $N = 2M + 1$, $\Delta k = \frac{2\pi}{N}\frac{1}{\Delta x}$. For $x_n = (-M + (n-1))\,\Delta x$, $n = 1, 2, \cdots, N$ we have

$$u(x_n, t) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} \tilde{u}(k_j, t) e^{ik_j x_n} \Delta k$$

where

$$\tilde{u}(k_j, t) = \frac{1}{\sqrt{2\pi}} \sum_{\tilde{n}=1}^{N} u(x_{\tilde{n}}, t) e^{-ik_j x_{\tilde{n}}} \Delta x$$

$k_j = (-M + (j-1))\,\Delta k$, $j = 1, \cdots, N$

a)

b)

c)

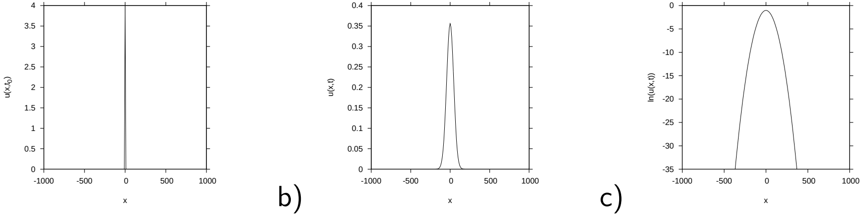


Figure: Ordinary diffusion.

$$\frac{\partial^2}{\partial x^2} u(x,t) = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^2} \Big(u(x-\Delta x, t) - 2u(x,t) + u(x+\Delta x, t) \Big) \quad .$$

$$\frac{1}{(\Delta x)^2}\frac{1}{\sqrt{2\pi}}\sum_{j=1}^{N}\tilde{u}(k_j,t)\left(e^{ik_jx_{n-1}}-2e^{ik_jx_n}+e^{ik_jx_{n+1}}\right)\Delta k$$
$$=\frac{1}{(\Delta x)^2}\frac{1}{\sqrt{2\pi}}\sum_{j=1}^{N}\tilde{u}(k_j,t)e^{ik_jx_n}(2\cos(k_j\Delta x)-2)\Delta k, \tag{12}$$

where $x_{n-1}=x_n-\Delta x$, $x_{n+1}=x_n+\Delta x$

$$\frac{\partial^2}{\partial x^2} u(x,t) = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^2} \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N} \tilde{u}(k_j, t) e^{ik_j x_n} (2\cos(k_j \Delta x) - 2) \Delta k.$$

For small k_j and for $f(k_j) = 2\cos(k_j \Delta x) - 2$ we have Taylor's expansion

$$f(k_j) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{d^\nu}{d{k_j}^\nu} f(k_j)|_{k_j=0} \cdot {k_j}^\nu = -{k_j}^2 (\Delta x)^2 + O({k_j}^4).$$

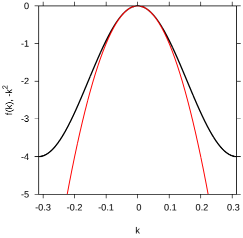


Figure: The Fourier kernel and its approximation by $-|k|^2$.

Generalization

$$\frac{\partial}{\partial t}u(x,t) = \chi\frac{\partial^\mu}{\partial x^\mu}u(x,t),\ \ \chi > 0$$

- $\mu = 2$–diffusion
- $0 < \mu < 2$–superdiffusion
- $\mu > 2$–subdiffusion

Fractional derivative

If $f(x) = x^m, \; m \in \mathbb{N}$, then for $n \in \mathbb{N}$, $m > n$ we have

$$\frac{d^n}{dx^n} x^m = m(m-1)\cdots(m-n+1)x^{m-n} = \frac{m!}{(m-n)!} x^{m-n}.$$

Using the Gamma function we can write

$$\frac{d^n}{dx^n} x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-n)} x^{m-n}.$$

For $\alpha > 0$ we put

$$\frac{d^\alpha}{dx^\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} x^{m-\alpha}.$$

$\alpha > 0$, $m < 0$

$$\frac{d^{\alpha}}{dx^{\alpha}} x^{m} = (-1)^{\alpha} \frac{\Gamma(-m+\alpha)}{\Gamma(-m)} x^{m-\alpha}$$

where $(-1)^{\alpha} = e^{i\alpha\pi}$

$$\frac{d^\alpha}{dx^\alpha}x^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}x^{m-\alpha} & for \quad m \geq 0, \\ e^{i\alpha\pi}\frac{\Gamma(-m+\alpha)}{\Gamma(-m)}x^{m-\alpha} & for \quad m < 0. \end{cases}$$

For any analytic function f

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

we extend the definition by

$$\frac{d^\alpha f}{dx^\alpha}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{d^\alpha}{dx^\alpha} x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)\, x^{k-\alpha}}{\Gamma(k+1-\alpha)}.$$

Riemann-Liouville integral $\alpha < 0$, $\alpha = -1$.

$$I_0 f(x) = \int_0^x f(t)\, dt.$$

Cauchy integral formula

$$I_0^n f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\, dt$$

for $n > 0$.

The (Riemann-Liouville) fractional integral is

$$I_0^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

for $\alpha > 0$.

The Riemann-Liouville fractional derivative of order $\alpha < 0$ is

$$D_0^{\alpha} = I_0^{-\alpha}$$

Property

$$f^{(m)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (it)^m \hat{f}(t) \exp(itx)\, dt$$

We define the Weyl derivative of order $\alpha \geq 0$ as

$$f^{(\alpha)} = \mathcal{F}^{-1}\left((it)^{\alpha} \mathcal{F} f\right) \tag{13}$$

The Riesz Derivative

$\mathbb{R}^n$

$$f^{(m)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (it)^m \hat{f}(t) \exp(itx)\, dt$$

$$\frac{\partial}{\partial x_k} f(\mathbf{x}) = \mathcal{F}^{-1}\left(\left[(it_k)\left[\mathcal{F}(f)\right](\mathbf{y})\right]\right)(\mathbf{x}),$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (t_1, \ldots, t_n)$, $k = 1, \ldots, n$.

$$-\Delta f(\mathbf{x}) = \mathcal{F}^{-1}\left(\left[|\mathbf{y}|^2 \left[\mathcal{F}(f)\right](\mathbf{y})\right]\right)(\mathbf{x})$$

The Riesz derivative of order α is defined as

$$I^\alpha f(\mathbf{x}) = \mathcal{F}^{-1}\left(\left[|\mathbf{y}|^\alpha \left[\mathcal{F}(f)\right](\mathbf{y})\right]\right)(\mathbf{x})$$

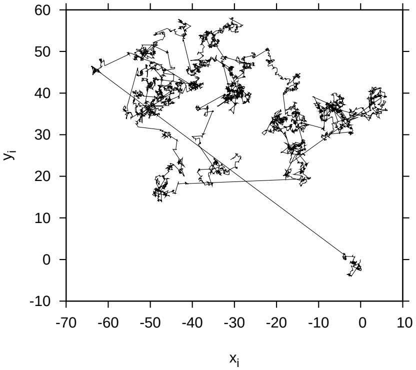
for $f \in \mathcal{S}(\mathbb{R}^n)$. Symbolically we have

$$I^\alpha = (-\Delta)^{\frac{\alpha}{2}}$$

or

$$\left(-\nabla^2\right)^{\frac{\alpha}{2}} f(\mathbf{x}) = \mathcal{F}^{-1}\left(\left[|\mathbf{y}|^\alpha \left[\mathcal{F}(f)\right](\mathbf{y})\right]\right)(\mathbf{x}). \qquad (14)$$

Lévy flight



Superdiffusion equation in Fourier space

$$\frac{\partial}{\partial t} u(x,t) = \chi \frac{\partial^\mu}{\partial x^\mu} u(x,t) \tag{15}$$

$$\frac{\partial}{\partial t} \tilde{u}(k,t) = -|k|^\mu \chi \tilde{u}(k,t) \tag{16}$$

$$\tilde{u}(k,t) = \tilde{u}(k,t_0) \mathrm{e}^{-|k|^\mu \chi (t-t_0)}. \tag{17}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k,t_0) \mathrm{e}^{-|k|^\mu \chi (t-t_0) + ikx} dk,$$

where

$$\tilde{u}(k,t_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(y,t_0) \mathrm{e}^{-iky} dy.$$

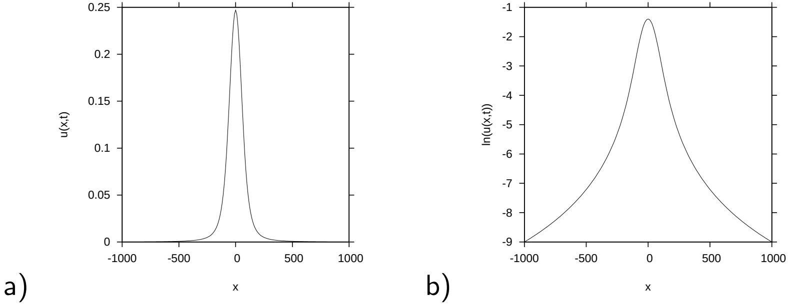


Figure: Superdiffusion for Lévy exponent 1.5. a) Density $u(x,t)$ over x for fixed time t. b) Logarithm of the density.

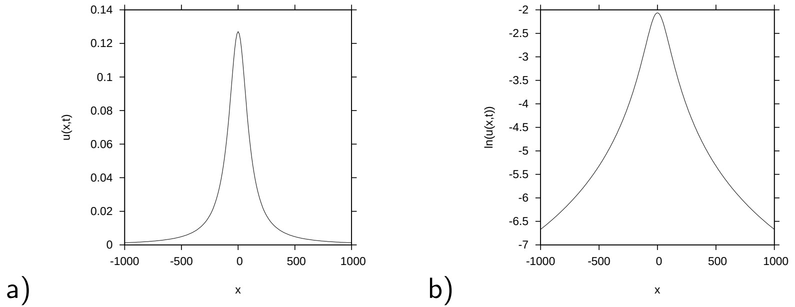


Figure: Superdiffusion for a lower Lévy exponent 1.0 a) Density $u(x, t)$ over x for fixed time t. b) Logarithm of the density.

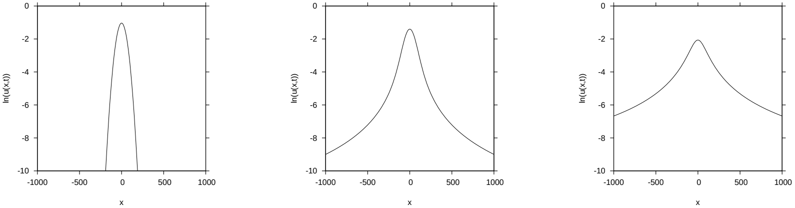


Figure: Comparison of the logarithm of density for a) Lévy exponent 2, b) Lévy exponent 1.5, c) exponent 1.0.

$$u(x,t) = \int_{-\infty}^{\infty} u(y,t_0)\, G(y-x, t-t_0)\, dy,$$

where

$$G(y-x, t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-ik(y-x)} \mathrm{e}^{-|k|^{\mu}\chi(t-t_0)}\, dk.$$

Substituting $z = \frac{y-x}{(\chi(t-t_0))^{\frac{1}{\mu}}}$ and $\tilde{k} = k(\chi(t-t_0))^{\frac{1}{\mu}}$ then we get

$$G(y-x, t-t_0) = \frac{1}{(\chi(t-t_0))^{\frac{1}{\mu}}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\tilde{k}z - |\tilde{k}|^{\mu}}\, d\tilde{k}.$$

Power law scaling of superdiffusion

$\mu = 2$

$$u(x,t) \sim e^{-x^2} \quad (18)$$

$\mu < 2$

$$u(x,t) \approx c \cdot |x|^{-(\mu+1)} \quad (19)$$

Lévy stable function

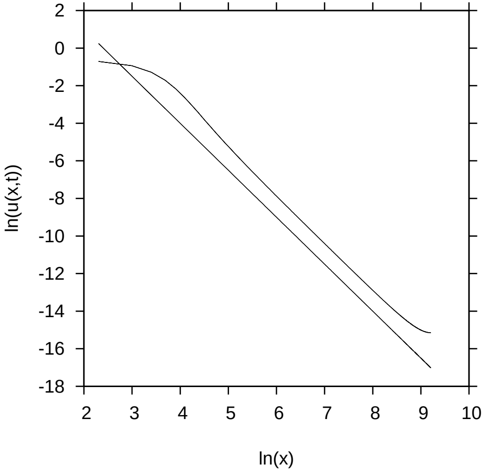
$$L_\mu(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-i\tilde{k}z - |\tilde{k}|^\mu} d\tilde{k} \tag{20}$$

Asymptotically, the stable law is

$$L_\mu(z) \sim \frac{1}{|z|^{1+\mu}}, \ |z| >> 1$$

In fact, a series expansion of L_μ gives us

$$L_\mu(z) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} z^{-(\mu k+1)} \Gamma(\mu k + 1) \sin\left(\frac{\pi}{2}\mu k\right) . \tag{21}$$


2
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
ln(u(x,t))
2 3 4 5 6 7 8 9 10
ln(x)

Diffusion in $\mathbb{R}^r$

$$\frac{\partial}{\partial t} u(\underline{x}, t) = \chi \sum_{i=1}^{r} \frac{\partial^2}{\partial {x_i}^2} u(\underline{x}, t) \tag{22}$$

where $\underline{x} = (x_1, \cdots, x_r)$.

$$\frac{\partial}{\partial t} \tilde{u}(\underline{k}, t) = -|\underline{k}|^2 \chi \tilde{u}(\underline{k}, t),$$

where $\underline{k} = (k_1, \cdots, k_r)$, $|\underline{k}|^2 = {k_1}^2 + \cdots + {k_r}^2$,

$$\tilde{u}(\underline{k}, t) = \frac{1}{(\sqrt{2\pi})^r} \int_{R^r} u(\underline{x}, t) e^{-i\underline{k}\,\underline{x}} \mathrm{d}\underline{x}$$

$$u(\underline{x},t) = \frac{1}{(4\pi\chi(t-t_0))^{\frac{r}{2}}} \int\limits_{\mathbb{R}^r} u(\underline{y},t_0) \exp\left\{ -\frac{|\underline{x}-\underline{y}|^2}{4\chi(t-t_0)} \right\} \mathrm{d}\underline{y};$$

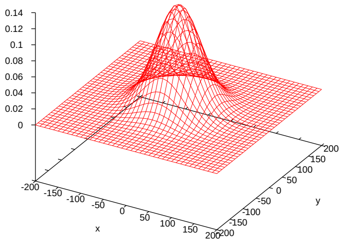
$$G(\underline{x}-\underline{y},t) = \frac{1}{(4\pi\chi(t-t_0))^{\frac{r}{2}}} \exp\left\{ -\frac{|\underline{x}-\underline{y}|^2}{4\chi(t-t_0)} \right\}$$

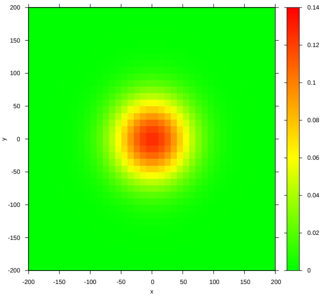
Case: $r = 2$. Let $N = 2M + 1$, $x_n = (-M + (n-1))\,\Delta x$, $y_m = (-M + (m-1))\,\Delta y$, $n, m = 1, 2, \cdots, N$, $\Delta k = \frac{2\pi}{N}\frac{1}{\Delta x}$, $\Delta \kappa = \frac{2\pi}{N}\frac{1}{\Delta \ell}$. Then we have

$$\tilde{u}(k_j, \kappa_\ell, t) = \frac{1}{2\pi}\sum_{n=1}^{N}\sum_{m=1}^{N} u(x_n, y_m, t)e^{-i\left(k_j x_n + \kappa_\ell y_m\right)}\Delta x \Delta y$$

where $j, \ell = 1, \ldots, N$ and the back Fourier transform

$$u(x_n, y_m, t) = \frac{1}{2\pi}\sum_{j=1}^{N}\sum_{\ell=1}^{N} \tilde{u}(k_j, \kappa_\ell, t)e^{i\left(k_j x_n + \kappa_\ell y_m\right)}\Delta k \Delta \kappa$$


0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-200 -150 -100 -50 0 50 100 150 200
x
-200 -150 -100 -50 0 50 100 150 200
y


200 150 100 50 0 -50 -100 -150 -200
y
-200 -150 -100 -50 0 50 100 150 200
x
0.14 0.12 0.1 0.08 0.06 0.04 0.02 0

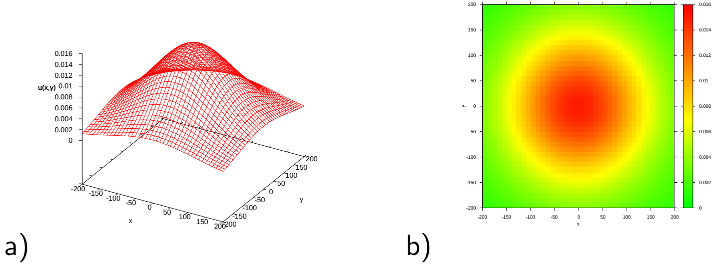


Figure: Superdiffusion for Lévy exponent 1.5. a) Density $u(x,t)$ over x for fixed time t. b)color-coded projection of $u(x,t)$

Integral representation of the fractional Laplacian

$$\frac{\partial^{\mu}}{\partial x^{\mu}} u(x,t) = \frac{1}{\pi}\Gamma(\mu+1)\sin\left(\frac{\pi}{2}\mu\right)\int_{-\infty}^{\infty}\frac{u(y,t)}{|x-y|^{\mu+1}}\mathrm{d}y \qquad (23)$$

The sketch of the proof

$$f(x) = (-ix)^{-\alpha} \quad \longleftrightarrow \quad \tilde{f}(k) = \frac{\sqrt{2\pi}}{\Gamma(\alpha)} H(k) k^{\alpha-1} \tag{24}$$

$$f(x) = (ix)^{-\alpha} \quad \longleftrightarrow \quad \tilde{f}(k) = \frac{\sqrt{2\pi}}{\Gamma(\alpha)} H(-k)(-k)^{\alpha-1} \tag{25}$$

$$\Gamma(\alpha) = \int\limits_0^\infty x^{\alpha-1}\, e^{-x}\, dx \tag{26}$$

For $z \in \mathbf{C}$ holds

$$\oint_C z^{\alpha-1} \cdot e^{-z} \, dz = 0 \tag{27}$$

along any contour C not including any pole of $f(z)$.

$$g(x) = -\frac{2}{\sqrt{2\pi}} \Gamma(\mu+1) |x|^{-(\mu+1)} \sin\left(\frac{\pi}{2}\mu\right) \quad \longleftrightarrow \quad \tilde{g}(k) = |k|^{\mu}$$

$$\frac{\partial^\mu}{\partial x^\mu} u(x,t) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} -|k|^\mu \tilde{u}(k,t) e^{ikx} \mathrm{d}k$$

The convolution theorem

$$h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy \tag{28}$$

then $\tilde{h}(x) = \sqrt{2\pi}\ \tilde{f}(k) \cdot \tilde{g}(k)$

In our case:

$$h(x) = \frac{\partial^\mu}{\partial x^\mu} u(x,t)$$

,

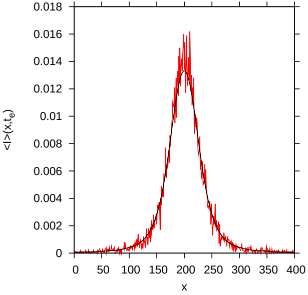
$$\tilde{h}(k) = -|k|^\mu \tilde{u}(k,t)$$

From this $h = -\frac{1}{\sqrt{2\pi}} u * g$, where $\tilde{g}(k) := |k|^\mu$. Then $g(x) = -\frac{2}{\sqrt{2\pi}} \Gamma(\mu+1)|x|^{-(\mu+1)} \sin\left(\frac{\pi}{2}\mu\right)$.

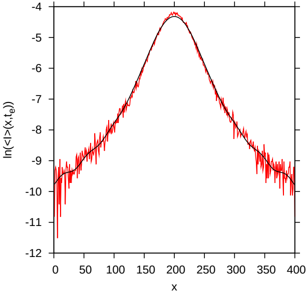
Random walk model

$$
\begin{aligned}
S_i + I_j &\xrightarrow{\chi} I_i + S_j \\
I_i + S_j &\xrightarrow{\chi} S_i + I_j
\end{aligned}
$$

$$
\chi c_\mu J_{ij} I_j \longrightarrow \frac{\chi c_\mu I_j}{\| x_i - x_j \|^{\mu+1}}
$$


0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
<I>(x,t_e)
0 50 100 150 200 250 300 350 400
x

a)


-4
-5
-6
-7
-8
-9
-10
-11
-12
ln(<I>(x,t_e))
0 50 100 150 200 250 300 350 400
x

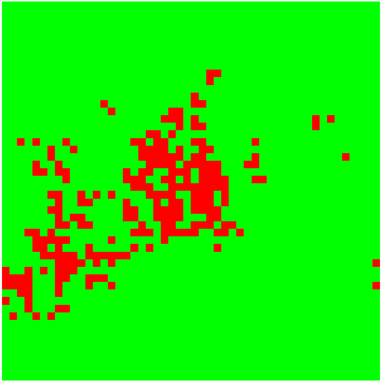
b)

SIS model

$$S_i + I_j \xrightarrow{\beta} I_i + I_j$$
$$I_i \xrightarrow{\alpha} S_i$$

$$\beta J_{ij} I_j \longrightarrow \frac{\beta I_j}{\| x_i - x_j \|^{\mu+2}}$$

$\mu = 2$



$\mu = 1.5$

Thank you for your attention